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Sequences of numbers and their associated functions

Mihai Pascu

Universitatea Petrol-Gaze din Ploieşti, Bd. Bucureşti 39, Ploieşti, Catedra de Matematicã
Institutul de Matematică al Academiei Române
e-mail: mpascu@upg-ploiesti.ro, Mihai.Pascu@imar.ro

Abstract

The relationship between the logarithmic convex sequences of real numbers and their associated functions is investigated. A convex function N *is constructed starting from a logarithmic sequence and then a function* $\widetilde{M}$ *is associated to* N. *We shall prove that these two functions are related through a Clairaut equation.*

Keywords: *logarithmic convex sequences, associated functions, Legendre transform*

One knows that a function $f : I(=(a,b)) \to \boldsymbol{R}$ is analytic if and only if for every compact set K contained in I there exist constants $C = C(K),\ A = A(K)$ so that

$$\left| f^{(p)}(x) \right| \le CA^p p!,\ (\forall) x \in K .$$

By applying Stirling formula one can see that a function is analytic if and only if for every compact set K contained in I there exist constants $C = C(K),\ A = A(K)$ so that

$$\left| f^{(p)}(x) \right| \le CA^p p^p,\ (\forall) x \in K .$$

Gevrey introduced the following classes of functions: a function $f : I(=(a,b)) \to \boldsymbol{R}$ belongs to a Gevrey class of order $\gamma > 1$ if for every compact set K contained in I there exist constants $C = C(K),\ A = A(K)$ so that

$$\left| f^{(p)}(x) \right| \le CA^p (p!)^\gamma,\ (\forall) x \in K$$

or, equivalently, if and only if for every compact set K contained in I there exist constants $C = C(K),\ A = A(K)$ so that

$$\left| f^{(p)}(x) \right| \le CA^p p^{p\gamma},\ (\forall) x \in K .$$

An essential difference between the space of analytic functions and Gevrey classes is that there is no analytic function with compact support, but there are enough Gevrey functions of order r with compact support. This allowed to develop the theory of Gevrey ultradistributions, following Schwartz's approach. Even more general spaces of functions (Denjoy-Carleman spaces) can be obtained if in the inequalities defining the Gevrey classes one replaces the sequence $(p^{p\gamma})_p$ with an arbitrary sequence of real numbers $(M_p)_p$. Ultradistributions corresponding to such classes of functions were studied in [3], [7]. Subclasses of the Schwartz space of rapidly decreasing functions (*S*-spaces) can also be obtained by using sequences $(M_p)_p$ of real numbers to control the behavior of the function and of its derivatives at infinity ([2], [7]). *S*-classes are appropriate for the study of parabolic equations and, in particular, of the heat equation [2]. Mandelbrojt [4] (in general) and Roumieu [7] (in the context of the theory of *S* functions) emphasized the importance of the function associated to a sequence of numbers.

In order to have good properties for the Denjoy-Carleman spaces of functions some minimal conditions on the sequences $(M_p)_p$ have to be imposed. In what follows we shall suppose that the sequence $\{M_p\}_p$ is logarithmic convex, i.e.

$$M_p^2 \le M_{p-1}M_{p+1},\ (\forall)p \ge 1. \tag{1}$$

Without restricting the generality, we may suppose that $M_0 = 1$. Therefore, if $\{M_p\}_p$ is logarithmic convex, then

$$M_pM_q \le M_0M_{p+q} = M_{p+q},\ (\forall)p,q \ge 0.$$

We shall also suppose that

$$\lim_{p\to\infty}\frac{M_p}{M_{p+1}} = 0. \tag{2}$$

For a logarithmic convex sequence $\{M_p\}_p$ we can define its associated function through the formula

$$M(r) = \sup_{p\ge 0}(p\ln r - \ln M_p),\ (\forall)r > 0. \tag{3}$$

It is well known that M is nondecreasing, $M(r) = 0$ for $r < M_1$ and that

$$\log M_p = \sup_{r>0}(p\ln r - M(r)).$$

In this paper we shall investigate the relationship between logarithimic convex sequences and their associated functions. More precisely, we shall construct a convex function N starting from a logarithmic convex sequence and we shall associate to N another function $\widetilde{M}$ through a formula similar to formula (2). We shall see that these two functions are related through an implicit differential equation (a Clairaut equation) and we shall compare $\widetilde{M}$ with M. We shall also put into evidence the link between $\widetilde{M}$ and the Legendre transform. Finally, we shall consider the case of Gevrey classes $\{M_p\}_p = (p^{p\gamma})_p$.

Proposition 1. Let $\{M_p\}_p$ be a sequence of real numbers as above and $N_p = \ln M_p$. Then there exists a continuously differentiable convex function $N:[0,\infty)\to[0,\infty)$ so that $N(p) = N_p$, $(\forall)p \in \mathbf{N}^*$, $N(0) = 0$ and $N'(s)\to\infty$ when $s\to\infty$.

Proof. Let us first remark that

$$0 \le N_p - N_{p-1} \le N_{p+1} - N_p,\ (\forall)p \in \mathbf{N}^*$$

and that

$$N_p - N_{p-1} \to \infty \text{ when } p \to \infty .$$

In order to prove that a function N with the required properties exists, it is sufficient to construct for any p, any $b \le N_{p+1} - N_p$ and some $c \in [N_{p+1} - N_p, N_{p+2} - N_{p+1}]$ (a choice for c will be made later) a function f having the following properties:

$$f(p) = N_p ,$$
$$f(p+1) = N_{p+1},$$
$$f'(p) = b,$$
$$f'(p+1) = c$$

and f' is increasing. If we put $g = f'$, one can see that it is sufficient to determine g so that

(a) $g(p) = b,\ g(p+1) = c$,
(b) g is increasing,
(c) $\int_p^{p+1} g(s)\mathrm{d}s = N_{p+1} - N_p$.

(If such a function g is determined, we can take $f(s) = N_p + \int_p^s g(t)\mathrm{d}t,\ (\forall)s \in [p, p+1]$).

We can take g of the form $g(s) = \alpha s^2 + \beta s + \gamma$. Conditions (a) and (c) are fulfilled if and only if

$$\begin{cases} \alpha p^2 + \beta p + \gamma = b \\ \alpha(p+1)^2 + \beta(p+1) + \gamma = c \\ \alpha(p^2 + p + \frac{1}{3}) + \beta(p + \frac{1}{2}) + \gamma = N_{p+1} - N_p \end{cases} .$$

Therefore we must have

$$\alpha = 3(c - N_{p+1} + N_p)$$

and

$$\beta = c - b - 3(2p+1)(c - N_{p+1} + N_p) .$$

In order to have condition (b) also satisfied, we have to select c so that

$$c \le \frac{3(N_{p+1} - N_p) - b}{2} .$$

This is always possible, since $\frac{3(N_{p+1} - N_p) - b}{2} \ge N_{p+1} - N_p$ for any p.

Remark. The space of functions defined by using some sequence $(M_p)_p$ remains the same if we change a finite number of terms of the sequence. Therefore when working with such spaces, one can always assume that $1 = M_0 \le M_1$. From the logarithmic convexity, it will follow that the sequence $(M_p)_p$ is increasing. In this case, the construction from the proof of Proposition 1 will give an increasing function *N*.

Sometimes it is useful to replace logarithmic convex sequences with sequences that are strictly logarithmic convex, i.e sequences which satisfy

$$M_p^2 < M_{p-1}M_{p+1},\ (\forall)p \geq 1. \tag{1'}$$

The following remark and Proposition 2 from below will show that there is no loss in generality in assuming that the sequences we are working with are strictly logarithmic convex.

Definition 1. Two sequences of real numbers $(M_p)_p$ and $(M'_p)_p$ are said to be equivalent if there exist two positive constants *c* and *C* so that

$$c^p M_p \leq M'_p \leq C^p M_p,\ (\forall)p \in \mathbf{N}^*.$$

Remark. The spaces of functions defined by equivalent sequences are equal.

Proposition 2. If $(M_p)_p$ is a logarithmic convex sequence which satisfies (2), then there exists a sequence $(M'_p)_p$ which is strictly logarithmic convex and is equivalent with $(M_p)_p$.

Proof. Suppose $p_1,\ p_1+n$ satisfy

$$M_{p_1}^2 < M_{p_1-1}M_{p_1+1},\ M_{p_1+n}^2 < M_{p_1+n-1}M_{p_1+n+1}$$

and

$$M_p^2 = M_{p+1}M_{p-1},\ (\forall)p \in \{p_1+1,\dots,p_1+n-1\}.$$

(Due to condition (2), we can not have an infinite sequence of consecutive terms M_p so that $M_p^2 = M_{p+1}M_{p-1}$.) Let us put, as above $N_p = \ln M_p$. Then

$$N_p - N_{p-1} = N_{p+1} - N_p,\ (\forall)p \in \{p_1+1,\dots,p_1+n-1\}.$$

Suppose we determined N'_p for $p \leq p_1$ so that

$$N'_p - N'_{p-1} < N'_{p+1} - N'_p,\ (\forall)p < p_1,\ N'_{p_1} - N'_{p_1-1} < N_{p_1+1} - N'_{p_1}$$

and

$$N_p \leq N'_p \leq \mathrm{e}^p N_p,\ (\forall)p \leq p_1.$$

It will be sufficient to determine N'_p for $p \in \{p_1+1,\dots,p_1+n\}$ such that

$$N'_p - N'_{p-1} < N'_{p+1} - N'_p,\ (\forall)p \in \{p_1+1,\dots,p_1+n-1\}, \tag{4}$$

$$N'_{p_1+n} - N'_{p_1+n-1} < N_{p_1+n+1} - N'_{p_1+n} \tag{5}$$

and

$$N_p \leq N'_p \leq p + N_p,\ (\forall)p \in \{p_1+1,\dots,p_1+n\}. \tag{6}$$

We shall use the following notations:

$$a = N_{p_1+2} - N_{p_1+1},$$

$$b = N_{p_1+n+1} - N_{p_1+n},$$

$$d = \min(b-a, n).$$

We put

$$N'_{p_1+j} = N_{p_1+1} + a(j-1) + \frac{(j-1)^2}{n^2}\cdot\frac{d}{2},\ (\forall)j \in \{1,...,n\}.$$

Then, clearly, (4) is fulfilled. Also, one can easily notice that (5) is true if $n^2 - 2n + 3 > 0,\ (\forall)n > 0$. Since this last inequality is true, we have also (5). Finally, the first inequalities from (6) are evident, from the construction of N'_p and

$$N'_{p_1+j} - N_{p_1+j} = \frac{(j-1)^2}{n^2}\cdot\frac{d}{2} \le j \le p_1 + j.$$

The proof has finished.

Starting from now we shall assume that the sequence $(M_p)_p$ is strictly logarithmic convex. Let us remark that in this case the function *N* from Proposition 1 can also be supposed to be a strictly convex function. So in what follows we shall always assume that $N : [0,\infty) \to [0,\infty)$ is a continuously differentiable strictly convex function so that

$$N(p) = N_p,\ (\forall)p \in \mathbf{N^*},\ N(0) = 0$$

and $N'(s) \to \infty$ when $s \to \infty$. We define a function $\widetilde{M}$ through a formula similar to formula (3)

$$\widetilde{M} : (0,\infty) \to [0,\infty),\ \widetilde{M}(r) = \sup_{s\ge 0}(s\ln r - N(s)),\ (\forall)r > 0. \tag{3'}$$

Proposition 3. The function $\widetilde{M}$ is correctly defined,

$$M(r) \le \widetilde{M}(r) \le M(r) + \ln r,\ (\forall)r > 0 \tag{7}$$

and $\widetilde{M}(r) > 0$ if and only if $\ln r > \inf_{s>0}\frac{N(s)}{s} = \inf_{s>0} N'(s)$.

Proof. The last assertion is obvious from the properties of *N*. Choose *r* so that $\ln r > \inf_{s>0} N'(s)$. We define a function

$$f_r : [0,\infty) \to \boldsymbol{R},\ f_r(s) = s\ln r - N(s),\ (\forall)s \ge 0.$$

Then $f'_r(s) = \ln r - N'(s)$. Since *N'* is strictly increasing and $N'(s) \to \infty$ when $s \to \infty$, there exists a unique point *s*(*r*) so that $f'_r(s(r)) = 0$. We have

$$\widetilde{M}(r) = f_r(s(r)) = s(r)\ln r - N(s(r)).$$

The first inequality from (7) is clearly a consequence of the definitions of *M* and $\widetilde{M}$. Given *p,* so that $s(r) \in [p, p+1]$ one obtains

$$\widetilde{M}(r) - M(r) \le s(r)\ln r - N(s(r)) - (p\ln r - N(p)) =$$
$$= (s(r) - p)\ln r - (N(s(r)) - N(p)) \le (s(r) - p)\ln r \le \ln r.$$

Lemma 1. If $f:[a,\infty)\to[0,\infty)$, is a continuous, strictly increasing and surjective function and if $F(t)=\int_0^t f^{-1}(s)\mathrm{d}s$, $(\forall)s\geq 0$, then the function

$$G:[a,\infty)\to\mathbf{R},\ G(s)=sf(s)-F(f(s))$$

is increasing.

Proof. For $s_2>s_1$ we have

$$G(s_2)-G(s_1)=s_2f(s_2)-s_1f(s_1)-\int_{f(s_1)}^{f(s_2)}f^{-1}(s)\mathrm{d}s\,.$$

If we put $t_2=f(s_2),\ t_1=f(s_1)$ we will notice that

$$G(s_2)-G(s_1)=t_2f^{-1}(t_2)-t_1f^{-1}(t_1)-\int_{t_1}^{t_2}f^{-1}(s)\mathrm{d}s\,.$$

The mean value property implies that there exists some t in the interval $[t_1,t_2]$ so that $\int_{t_1}^{t_2}f^{-1}(s)\mathrm{d}s=(t_2-t_1)f^{-1}(t)$. Therefore

$$G(s_2)-G(s_1)=t_2f^{-1}(t_2)-t_1f^{-1}(t_1)-(t_2-t_1)f^{-1}(t)=$$
$$=t_2(f^{-1}(t_2)-f^{-1}(t))+(t_2-t_1)f^{-1}(t)+t_1(f^{-1}(t)-f^{-1}(t_1))-(t_2-t_1)f^{-1}(t)>0\,.$$

The proof of the lemma has finished.

Proposition 4. The function $\widetilde{M}$ is strictly increasing.

Proof. It is suficient to prove that $\widetilde{M}\circ\exp$ is increasing. But

$$\widetilde{M}(\mathrm{e}^t)=t(N')^{-1}(t)-N\circ(N')^{-1}(t)\,.$$

We can apply Lemma 1 with $f=(N')^{-1}$.

Proposition 5. The function N satisfy the differential equation:

$$(\widetilde{M}\circ\exp)(p)=p\tau-x\,,$$

where, as usually, we denoted with p the derivative of the dependent variable x with respect to the independent variable τ.

Proof. We saw that we have $\widetilde{M}(r)=s(r)\ln r-N(s(r))$ where $s(r)=(N')^{-1}(\ln r)$. If we make the change of variables $\ln r=t$, we obtain the equality

$$\widetilde{M}(\mathrm{e}^t)=t(N')^{-1}(t)-N\circ(N')^{-1}(t)\,.$$

We obtain our equation if we make a second change of variables: $\tau=(N')^{-1}(t)$.

We shall investigate now the relation between $\widetilde{M}$ and the Legendre transform. Let us recall that if L is a convex function defined on the whole real axis so that

$$\lim_{|s|\to\infty}\frac{L(s)}{|s|}=\infty\,,$$

then one can define its Legendre transform through the formula $L^*(t) = \sup_{s\in R}(st - L(s))$.

The Legendre transform L^* is also a superlinear convex function and $(L^*)^* = L$.

In our case, the function N is a convex function defined only on the positive semiaxis. But if we assume that N is increasing (we saw that we can always assume this), then we can extend N by parity to a convex function $\widetilde{N}$ defined on the whole real axis. More than that, due to the fact that $N'(s) \to \infty$ when $s \to \infty$, we see that $\lim_{|s|\to\infty} \frac{\widetilde{N}(s)}{|s|} = \infty$.

Therefore we can define the Legendre transform of $\widetilde{N}$. We can easily see that

$$(\widetilde{M} \circ \exp)(t) = \widetilde{N}^*(|t|),\ (\forall)t \in \boldsymbol{R}.$$

Hence $\widetilde{M}(r) = \widetilde{N}^*(\ln r)$ for ln r > 0. We could use this relation to prove that $\widetilde{M}$ is correctly defined. But the differentiability of N allowed us to give a more precise expression for $\widetilde{M}$ and to obtain the result from Proposition 5.

As a consequence of these remarks, we have

$$\lim_{r\to\infty} \frac{\widetilde{M}(r)}{\ln r} = \infty,$$

and, according to (7) $\widetilde{M}$ and M have the same order of magnitude i.e. $(\exists)C > 0$ so that

$$M(r) \le \widetilde{M}(r) \le CM(r),\ (\forall)r > 0.$$

(We could also deduce the inequality from above using the well known fact that $\lim_{r\to\infty} \frac{M(r)}{\ln r} = \infty$.)

Let us consider the case $\{M_p\}_p = (p^{p\gamma})_p$. In this situation $N_p = \gamma p \ln p$ and we can take $N(s) = \gamma s \ln s$. Therefore $f'_r(s) = \ln r - \gamma \ln s - \gamma$ and $s(r) = \mathrm{e}^{-1} r^{1/\gamma}$. A direct computation shows that $\widetilde{M}(r) = \gamma \mathrm{e}^{-1} r^{1/\gamma}$. The upper estimate obtained in Proposition 3 can be improved. Indeed,

$$\begin{aligned} s(r)\ln r - N(s(r)) - (p\ln r - N(p)) &= s(r)\ln r - \gamma s(r)\ln s(r) - (p\ln r - \gamma p\ln p) \le \\ &\le s(r)\ln r - \gamma s(r)\ln s(r) - (p\ln r - \gamma p\ln s(r)) = \\ &= (s(r) - p)\ln r - \gamma(s(r) - p)(\gamma^{-1}\ln r - \ln \mathrm{e}) = \\ &= \gamma(s(r) - p) \le \gamma. \end{aligned}$$

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Şiruri de numere şi funcţiile lor asociate

Rezumat

Este investigată legătura dintre şirurile de numere logaritmic convexe şi funcţiile lor asociate. Se construieşte o funcţie convexă N *pornind de la un şir logaritmic, după care se asociază lui* N *o funcţie* $\widetilde{M}$ *. Vom demonstra că aceste două funcţii sunt dependente printr-o ecuaţie Clairaut.*