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The Approximate Calculation of Definite Simple and Multiple Integrals

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Abstract

This paper presents the logical schemata and the programs in turbo-pascal for the approximate calculation methods of the simple definite integral [2], of the double definite integral [3], and of the triple definite integral [4]. At the end there are analized the calculation errors corresponding to these methods that use the polynomials of interpolation to five nodes.

Key words: *division, interpolation, algorithm*

Introduction

Paper [2] presents method "B" for the approximate calculation of simple definite integral

$$I_1 = \int_a^b f(x)dx \tag{1}$$

where $f:[a,b]\to R$ is a integrable function on $[a,b]$.

It is considered an equidistant division with $4 \cdot n$ nodes.

$$\Delta : (x_0 = a < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{4 \cdot n} = b)$$

with the distance of the division $h = \dfrac{b-a}{4n}$, then the approximate value of the integral I_1 dimensions:

$$\bar{I}_1 = \frac{2 \cdot h}{45}\left[7(f(a)+f(b)) + 14\left(\sum_{k=1}^{n-1} f(x_{4k})\right) + 32\left(\sum_{k=0}^{n-1} f(x_{4k+1})\right) + \right.$$
$$\left. + 12\left(\sum_{k=0}^{n-1} f(x_{4k+2})\right) + 32\left(\sum_{k=0}^{n-1} f(x_{4k+3})\right) \right] \tag{2}$$

for which the calculation error is:

$$e_\tau = \left| I_1 - \bar{I}_1 \right| \le \frac{4 \cdot M \cdot (b-a)}{15 \cdot 21} \cdot h^6 \tag{3}$$

where:

$$M = \sup_{x\in[a,b]} \left| f_{(x)}^{(5)} \right|, \text{ cu } f \in C_{[a,b]}^{5}.$$

In paper [3] the "BxB" method describes the approximate calculation of the double definite integral:

$$I_2 = \iint_D f(x,y)dxdy \tag{4}$$

with $f : D \to R$, integrable, where $D \subset R^2$ and it is bordered by both $y = y_1(x)$ with $y = y_2(x)$ with $y_1(x) \le y_2(x)$ for any $x \in [a,b]$.

The approximate value of the integral I_2 is given by the formula:

$$\bar{I}_2 = \frac{2\cdot h}{45}\left[7(\varphi(x_0)+\varphi(x_{4\cdot n}))+14\left(\sum_{k=1}^{n-1}\varphi(x_{4\cdot k})\right)+32\left(\sum_{k=0}^{n-1}\varphi(x_{4k+1})\right)+12\left(\sum_{k=0}^{n-1}\varphi(x_{4k+2})\right)+\right.$$

$$\left.+32\left(\sum_{k=0}^{n-1}\varphi(x_{4k+3})\right)\right] \tag{5}$$

where:

$$\varphi(x_i) = \frac{2\cdot h_1}{45}\left[7(f(x_i, y_0)+f(x_i, y_{4\cdot m}))+14\left(\sum_{k=1}^{m-1} f(x_i, y_{4k})\right)+32\left(\sum_{k=0}^{m-1} f(x_i, y_{4k+1})\right)+\right.$$

$$\left.+12\left(\sum_{k=0}^{m-1} f(x_i, y_{4k+2})\right)+32\left(\sum_{k=0}^{m-1} f(x_i, y_{4k+3})\right)\right] \tag{6}$$

where the norm of the interval [a, b] is: $h = \frac{b-a}{4n}$,and the norm of the interval $[y_1(x_i), y_2(x_i)]$ is: $h_1 = \frac{y_2(x_i) - y_1(x_i)}{4\cdot m}$ with $i \in \{0,1,...4\cdot n\}$.

Paper [4] presents the method "BxBxB" for the approximate calculation of the triple definite integrals.

$$I_3 = \iiint_V f(x,y,z)dxdydz \tag{7}$$

with $f : V \to \mathbf{R}$ integrable on $V \subset R^3$. The domain V is defined by surfaces: $z = z_1(x,y)$ and $z = z_2(x,y)$ with $z_1(x,y) \le z_2(x,y)$ for $(\forall)(x,y) \in D$, where D is the projection of the domain V on the xOy plan and it is defined by the curves: $y = y_1(x)$ and $y = y_2(x)$, with $y_1(x) \le y_2(x)$ for $(\forall)x \in [a,b]$, where [a, b] is the projection of D on the axis Ox.

The approximate value of the integral I_3 is given by the formula:

$$\bar{I}_3 = \frac{2\cdot h}{45}\left[7(\varphi(x_0)+\varphi(x_{4\cdot n}))+14\left(\sum_{k=1}^{n-1}\varphi(x_{4\cdot k})\right)+32\left(\sum_{k=1}^{n-1}\varphi(x_{4k+1})\right)+12\left(\sum_{k=0}^{n-1}\varphi(x_{4k+2})\right)+\right.$$

$$\left.+32\left(\sum_{k=0}^{n-1}\varphi(x_{4k+3})\right)\right] \tag{8}$$

where the interval [a, b] has the norm $h = \nu(\Delta_1) = \frac{b-a}{4\cdot n}$, and

$$\varphi(x_i) = \frac{2 \cdot h_1}{45}\left[7\left(g(x_i, y_0) + g(x_i, y_{4\cdot m})\right) + 14\left(\sum_{k=1}^{m-1} g(x_i, y_{4k})\right) + 32\left(\sum_{k=1}^{m-1} g(x_i, y_{4k+1})\right) + \right.$$

$$\left. + 12\left(\sum_{k=0}^{m-1} g(x_i, y_{4k+2})\right) + 32\left(\sum_{k=0}^{m-1} g(x_i, y_{4k+3})\right)\right] \qquad (9)$$

where the interval $[y_1(x_i), y_2(x_i)]$ has the norm $h_1 = \nu(\Delta_2) = \frac{y_2(x_i) - y_1(x_i)}{4 \cdot m}$, and

$$g(x_i, y_i) = \frac{2h_2}{45}\left[7\left(f(x_i, y_j, z_0) + f(x_i, y_j, z_{4\cdot p})\right) + 14\left(\sum_{k=1}^{p-1} f(x_i, y_j, z_{4\cdot k})\right) + \right.$$

$$\left. + 32\left(\sum_{k=0}^{p-1} f(x_i, y_j, z_{4k+1})\right) + 12\left(\sum_{k=0}^{p-1} f(x_i, y_j, z_{4k+2})\right) + 32\left(\sum_{k=0}^{p-1} f(x_i, y_j, z_{4k+3})\right)\right] \qquad (10)$$

for $i \in \{1,2,...,4 \cdot n\}$, $j \in \{0,1,2,...,4 \cdot m\}$, and the interval $[z_1(x_i, y_j), z_2(x_i, y_j)]$ has the norma of the equidistant division $h_3 = \nu(\Delta_3) = \frac{z_2(x_i, y_j) - z_1(x_i, y_j)}{4 \cdot p}$.

Contents

A. Programs

```
program simple_integral;
type vec=array[1..50] of real;
var a1:vec;
    a,b,ss,s,h,t:real;
    i,n:integer;

function sum(var a:vec; p:integer; hp:real):real;
var k:integer;
    s1,s2,s3,s4,ss:real;
begin
s1:=0;
s2:=0;
s3:=0;
s4:=0;
for k:=0 to p-1 do
   begin
   s1:=s1+a[4*k+1];
   s2:=s2+a[4*k+2];
   s3:=s3+a[4*k+3];
   s4:=s4+a[4*k+4]
   end;
s1:=s1-a[1];
ss:=7*(a[1] + a[4*n+1]);
ss:=ss+14*s1+32*S2+12*s3+32*s4;
ss:=(ss*2*hp)/45;
sum:=ss
end;
```

```
function fun(var i:integer; a,h:real):real;
var x1:real;
begin
x1:=a+i*h;
fun:=(x1*x1)/(1+x1*x1*x1)
end;

begin
{clrscr;}
write('a=');
readln(a);
write('b=');
readln(b);
write('n=');
readln(n);
h:=(b-a)/(4*n);
for i:=0 to 4*n do
  a1[i+1]:=fun(i,a,h);
s:=sum(a1,n,h);
writeln('Integral I=',s:12:6);
{repeat until keypressed}
end.
```

program double_integral;

```
type vec=array[1..50] of real;
var x,fi,c:vec;
    a,b,ss,h,a1,b1,h1:real;
    i,j,n,m:integer;

function sum(var a:vec;p:integer;hp:real):real;
var k:integer;
    s1,s2,s3,s4,ss:real;
begin
s1:=0;
s2:=0;
s3:=0;
s4:=0;
for k:=0 to p-1 do
  begin
  s1:=s1+a[4*k+1];
  s2:=s2+a[4*k+2];
  s3:=s3+a[4*k+3];
  s4:=s4+a[4*k+4]
  end;
s1:=s1-a[1];
ss:=7*(a[1] + a[4*n+1]);
ss:=ss+14*s1+32*S2+12*s3+32*s4;
ss:=(ss*2*hp)/45;
sum:=ss
end;

function fun2(i,j:integer; a,h,a1,h1:real):real;
var x1,y1:real;
begin
x1:=a+i*h;
y1:=a1+j*h1;
fun2:=1/sqrt(1+x1*x1+y1*y1)
end;
```

```
function y2x(var i:integer; a,h:real):real;
var x1,t:real;
begin
x1:=a+i*h;
t:=sqrt(abs(3-x1*x1));
y2x:=t;
end;

function y1x(var i:integer; a,h:real):real;
var x1,t:real;
begin
x1:=a+i*h;
t:=-sqrt(abs(3-x1*x1));
y1x:=t
end;

begin
{clrscr; }
write('a=');
readln(a);
write('b=');
readln(b);
write('n=');
readln(n);
write('m=');
readln(m);
h:=(b-a)/(4*n);
for i:=0 to 4*n do
  begin
  a1:=y1x(i,a,h);
  b1:=y2x(i,a,h);
  h1:=(b1-a1)/(4*m);
  for j:=0 to 4*m do
     c[j+1]:=fun2(i,j,a,h,a1,h1);
  fi[i+1]:=sum(c,m,h1);
  end;
ss:=sum(fi,n,h);
writeln('Integral I2=', ss:12:6)
end.
```

program treble_integral;

```
{uses crt;}
type vec=array[1..50] of real;
var c,g,fi:vec;
a,b,h,a1,b1,h1,a2,b2,h2,ss:real;
i,j,k,n,m,p:integer;

function sum(var a:vec;p:integer;hp:real):real;
var k:integer;
    s1,s2,s3,s4,ss:real;
begin
s1:=0;
s2:=0;
s3:=0;
s4:=0;
for k:=0 to p-1 do
  begin
  s1:=s1+a[4*k+1];
  s2:=s2+a[4*k+2];
```

```
  s3:=s3+a[4*k+3];
  s4:=s4+a[4*k+4]
  end;
s1:=s1-a[1];
ss:=7*(a[1] + a[4*n+1]);
ss:=ss+14*s1+32*S2+12*s3+32*s4;
ss:=(ss*2*hp)/45;
sum:=ss;
end;

function fun3(var i,j,k:integer; a,h,a1,h1,a2,h2:real):real;
var x1,y1,z1,t:real;
begin
x1:=a+i*h;
y1:=a1+j*h1;
z1:=a2+k*h2;
t:=sqrt(x1*x1+y1*y1+z1*z1);
fun3:=t
end;

function y2x(var i:integer; a,b:real):real;
var x1:real;
begin
x1:=a+i*b;
y2x:=sqrt(abs(9-x1*x1))
end;

function y1x(var i:integer; a,b:real):real;
var x1:real;
begin
x1:=a+i*b;
y1x:=-sqrt(abs(9-x1*x1))
end;

function z2x(var i,j:integer; a,h,a1,h1:real):real;
var x1,y1,t:real;
begin
x1:=a+i*h;
y1:=a1+j*h1;
t:=1;
z2x:=t
end;

function z1x(var i,j:integer; a,h,a1,h1:real):real;
var x1,y1:real;
begin
x1:=a+i*h;
y1:=a1+j*h1;
z1x:=(x1*x1+y1*y1)/9
end;

begin
{clrscr;}
write('a=');
readln(a);
write('b=');
readln(b);
write('n=');
readln(n);
```

```
write('m=');
readln(m);
write('p=');
readln(p);
h:=(b-a)/(4*n);
for i:=0 to 4*n do
  begin
  a1:=y1x(i,a,h);
  b1:=y2x(i,a,h);
  h1:=(b1-a1)/(4*m);
  for j:=0 to 4*m do
    begin
    a2:=z1x(i,j,a,b,a1,b1);
    b2:=z2x(i,j,a,b,a1,b1);
    h2:=(b2-a2)/(4*p);
    for k:=0 to 4*p do
       c[k+1]:=fun3(i,j,k,a,h,a1,h1,a2,h2);
    g[j+1]:=sum(c,p,h2);
    end;
  fi[i+1]:=sum(g,m,h1);
  end;
ss:=sum(fi,n,h);
writeln('Integral I2=' ,ss:12:6)
end.
```

B. Applications

Let us calculate the following definite integrals:

$$I_1 = \int_0^1 \frac{x^2 dx}{1+x^3};$$

$$I_2 = \iint_D \frac{dxdy}{\sqrt{1+x^2+y^2}} \text{, where } D = \{(x,y) \mid x^2 + y^2 \le 9\};$$

$$I_3 = \iiint_V (x^2+y^2+z^2)\,dxdydz \text{, where } V = \{(x,y,z) \mid x^2+y^2 \le 9z^2 \text{ and } z \in [0,1]\}$$

using the anterior programs.

Solution

Directly calculated by means of the analytical methods we obtain:

$$I_1 = \frac{\ell n\ 2}{2} = 0{,}231004906;$$

$$I_2 = 13{,}58599122$$

$$I_3 = 49{,}480084$$

Approximately calculated by means of "B", "BxB", "BxBxB" methods and using the programmes presented above we obtain:

$$\overline{I}_1 = 0{,}231049 \quad for \quad n = 2, a = 0, b = 1;$$

$$\overline{I}_2 = 13{,}585985 \quad for \quad n = m = 10, a = -3, b = 3;$$

$$\overline{I}_3 = 49{,}480048 \quad for \quad n = m = p = 10, a = -3, b = 3.$$

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Calculul aproximativ al integralelor definite simple şi multiple

Rezumat

În această lucrare sunt prezentate schemele logice şi programele în Turbo-Pascal pentru metodele de calcul aproximativ ale integralelor definite simple [2], ale integralelor definite duble [3] şi ale integralelor definite triple [4]. În final sunt analizate erorile de calcul corespunzătoare acestor metode care folosesc polinoamele de interpolare cu cinci noduri