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Interpolation of the Functions with Two Variable Values with Simple Nodes

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Abstract

The theme of interpolation of a function with 2 variable values is a complex matter which raises major difficulties when solving it according to its form, the grade of interpolation of the polynomial and the number of the points from the definition field. Each problem has its own way of solving which may represent a research topic.In this paper there are presented two general cases of interpolation with polynomials of the degre „n".

Key words: *interpolation, network, division, divided differences.*

Introduction

Considering a function $f:[a,b]\to \mathbf{R}$ and a division of the interval $[a,b]$:

$$\Delta:(x_1 = a < x_2 < x_3 < ... < x_n = b),$$

for which it is considered the point $A_k(x_k, y_k)$ with $y_k = f(x_k)$ for $k \in \{1,2,...,n\}$.

There should be determined a polynomial of the degree n named P which should approximate the function f on the interval $[a,b]$ so as:

$$P(x_k) = y_k = f(x_k) \text{ for } k \in \{1,2,...,n\}. \quad (1)$$

The answer to this problem is given by the formula of Lagrange [3]:

$$P(x) = \sum_{k=1}^{n} y_k \frac{Q_k(x)}{Q_k(x_k)}, \quad (2)$$

where

$$Q_k(x) = \frac{\omega(x)}{x - x_k} \text{ and } \omega(x) = \prod_{i=1}^{n}(x - x_i), \quad (3)$$

or by the formula of Newton [3] using the divided differences:

$$P(x)=P(f;x_1,x_2,...,x_n;x)=f[x_1]+f[x_1,x_2](x-x_1)+f[x_1,x_2,x_3]\cdot$$
$$\cdot(x-x_1)(x-x_2)+...+f[x_1,x_2,...,x_n](x-x_1)(x-x_2)...(x-x_{n-1}) \quad (4)$$

where the divided differences are:

$$f[x_1]=f(x_1) \; and \quad f[x_1,x_2,...,x_k]=\frac{f[x_1,...,x_{k-1}]-f[x_2,...,x_k]}{x_1-x_k}, \quad (5)$$

or

$$f[x_1,x_2,...,x_k]=\int_0^1 dt_1\left(\int_0^{t_1} dt_2\left(...\int_0^{t_{k-2}} f^{(k-1)}(x_1+t_1(x_2-x_1)+...+t_{k-1}(x_k-x_{k-1}))dt_{k-1}...\right)\right).(6)$$

In the case of a function of two variable values:

$$f:\Delta\to\mathbf{R},\ \Delta\subset\mathbf{R}^2$$

for which the points of interpolation $(x_i,y_i)$are known with $i\in\{0,1,...,n\}$ there should be determined a polynomial $P(x,y)$ $(x,y)\in\Delta$, and:

$$P(x_i,y_i)=f(x_i,y_i) \text{ for } i\in\{0,1,...,n\}. \quad (7)$$

In this case the problem of a function of two or more variable values, implies more difficulties.

First of all, if the interpolation polynomial has the form:

$$P_m(x,y)=a_{00}+(a_{10}\cdot x+a_{01}\cdot y)+\left(a_{20}\cdot x^2+a_{11}\cdot x\cdot y+a_{02}\cdot y^2\right)+...$$
$$+\left(a_{m0}\cdot x^m+a_{m-1,1}\cdot x^{m-1}\cdot y+...+a_{0m}\cdot y^m\right) \quad (8)$$

being a polynomial of degree m in the two variable values, then the coefficients of this polynomial should be determined from the conditions (7). So as for the polynomial given by (8), $(m+1)(m+2)/2$ are unknown coefficients, and from (7) there results $n+1$ conditions, then

$$\frac{(m+1)(m+2)}{2}=n+1 \quad (9)$$

or

$$n=\frac{m(m+3)}{2}\ ,\ m,n\in\mathbf{N}. \quad (10)$$

There results that the interpolation points can't be chosen arbitrarily because the system (7) from which the coefficients of the polynomial are determined must be a uniquely determined compatible system.

Secondly, the determined system (7), when condition (10) is fulfilled, needs to differ from zero to have a unique solution. This may have different forms, depending on $m\in\mathbf{N}^*$. If $m=1$, then we have from (8):

$$P_1(x,y)=a_{00}+a_{10}\cdot x+a_{01}y, \quad (11)$$

and from (7) the following conditions are obtained:

$$\begin{cases} P_1(x_0,y_0)=f(x_0,y_0) \\ P_1(x_1,y_1)=f(x_1,y_1) \\ P_2(x_2,y_2)=f(x_2,y_2) \end{cases}, \quad (12)$$

which have the determinant of the linear system:

$$\Delta = \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}.$$

In order for the system to have a unique solution, we need to have:

$$\Delta \neq 0 \tag{13}$$

(condition) which is equivalent to the non-co-linearity of three points (x_0, y_0), (x_1, y_1) and (x_2, y_2).

Thirdly, establishing the error in the case of interpolation of a function of two variable values is much difficult than in the case of a real variable value.

Fourthly, the interpolation of the functions of two or more variable values depends on the definition field. This matter is very important.

For example in plan of a rectangular field with parallel edges to the axes of coordination, the used formulae of interpolation may expand in the case of the functions with a real variable value.

In the case of a triangular field, this solution is not convenient.

Besides, the number of operations rises multiplicatively to the number of nodes with two or more dimensions. There are presented some cases frequently met.

A. Considering

$$\Delta = \{(x, y) \mid a \leq x \leq b; c \leq y \leq d\},$$

for which we consider the divisions of the intervals [*a*,*b*] and [*c*,*d*] named Δ_1 and Δ_2:

$\Delta_1 : (x_0 = a < x_1 < x_2 < ... < x_n = b)$

$\Delta_2 : (y_0 = c < y_1 < y_2 < ... < y_m = d)$

$\Delta_3 = \Delta_1 \times \Delta_2$ division of the Δ field and the function:

$$f : \Delta \to \mathbf{R}$$

for which we know $f(x_i, y_j)$ with $i \in \{0,1,...,n\}$, $j \in \{0,1,...,m\}$, then there is a unique polynomial $P_{n,m}(x, y)$ so that:

$$P_{n,m}(x_i, y_j) = f(x_i, y_j), i \in \{0,1,...,n\}, \ j \in \{0,1,...,m\}, \tag{14}$$

with the degree $P_{n,m}(x, y)$ reported to *x* smaller than or equal to *n* and reported to *y* smaller than or equal to *m*.

The base of polynomials Lagrange for the nodes $\{x_0, x_1,..., x_n\}$ for $\{L_i(x)\}$with $i \in \{0,1,2,...,n\}$and for the nodes $\{y_0, y_1,..., y_m\}$ is $\{\overline{L}_j(y)\}$with $j \in \{0,1,...,m\}$, where:

$$L_i(x) = \frac{Q_i(x)}{Q_i(x_i)}; \ \overline{L}_j(y) = \frac{\overline{Q}_j(y)}{\overline{Q}_j(y_j)} \tag{15}$$

$$Q_i(x) = \frac{\omega(x)}{x - x_i};\ \overline{Q}_j(y) = \frac{\overline{\omega}(y)}{y - y_j} \tag{16}$$

$$\omega(x) = \prod_{i=0}^{n}(x - x_i); \quad \overline{\omega}(y) = \prod_{j=0}^{m} y - y_j. \tag{17}$$

These series of polynomials with the degree *n* and *m* have the following characteristics:

$$\begin{cases} L_i(x_k) = 0 \text{ and } L_i(x_i) = 1 \text{ for } k \neq i \text{ with } i, k \in \{0,1,...n\} \\ \overline{L}_j(y_\ell) = 0 \text{ and } \overline{L}_j(y_j) = 1 \text{ for } \ell \neq j \text{ with } j, \ell \in \{0,1,2,...,m\}. \end{cases} \tag{18}$$

Then the interpolation polynomial that modifies (14) has the following form:

$$P_{n,m}(x, y) = \sum_{k=0}^{n}\sum_{\ell=0}^{m} f(x_k, y_\ell)\cdot L_k(x)\cdot \overline{L}_\ell(y). \tag{19}$$

Taking this into consideration (18), it results:

$$P_{n,m}(x_i, y_j) = f(x_i, y_j) \text{ for } i \in \{0,1,...,n\}, \text{ and } j \in \{0,1,...,m\}.$$

The error that we make when replacing the function *f* by the interpolation polynomial (19) is [3]:

$$e_\tau = \left| f(x, y) - P_{n,m}(x, y) \right| \leq \frac{M}{n!\cdot m!}\cdot m_1 \cdot m_2, \tag{20}$$

where

$$\begin{cases} M = \max\limits_{(x,y)\in D} \dfrac{\partial^{n+m} f(x, y)}{\partial x^n \partial y^m}; \\ m_1 = \max\limits_{x\in[a,b]} \left|\omega(x);\right| \\ m_2 = \max\limits_{y\in[c,d]} \left|\overline{\omega}(y).\right| \end{cases}. \tag{21}$$

B. Considering

$$D = \{(x, y) \mid a \leq x \leq b; c \leq y \leq d\},$$

for which we take into account the divisions of the intervals [*a*,*b*] and [*c*,*d*] marked with Δ_1 and Δ_2:

$$\begin{aligned} &\Delta_1 : (x_0 = a < x_1 < x_2 < ... < x_n = b) \\ &\Delta_2 : (y_0 = c < y_1 < y_2 < ... < y_n = d), \end{aligned}$$

with $\Delta = \Delta_1 \times \Delta_2$ the division of the field *D* and the function

$$f : D \to \mathbf{R}$$

for which we use only $(n+1)(n+2)/2$ nodes of interpolation, that means we know the value of the function *f*:

$$\begin{cases} f(x_0,y_0), f(x_1,y_0),..., f(x_{n-1},y_0), f(x_n,y_0); \\ f(x_0,y_1), f(x_1,y_1),..., f(x_{n-1},y_1); \\ \vdots \\ f(x_0,y_{n-1}), f(x_1,y_{n-1}); \\ f(x_0,y_n). \end{cases} \tag{22}$$

Then, there is a unique interpolation polynomial of the degree n in the variable value x also y and has the form:

$$\begin{aligned} P_n(x,y) &= a_{0,0} + a_{1,0}(x-x_0) + a_{0,1}(y-y_0) + a_{2,0}(x-x_0)(x-x_1) + \\ &+ a_{1,1}(x-x_0)(y-y_0) + a_{0,2}(y-y_0)(y-y_1) + ... + a_{n,0}(x-x_0)(x-x_1)....\\ &...(x-x_{n-1}) + a_{n-1,1}(x-x_0)(x-x_1)...(x-x_{n-2})\cdot(y-y_0) + a_{n-2,2}(x-x_0)....\\ &...(x-x_{n-3})\cdot(y-y_0)(y-y_1) + ... + a_{0,n}(y-y_0)(y-y_1)........(y-y_{n-1}) \end{aligned} \tag{23}$$

The coefficients of this polynomial are determined by means of the next conditions:

$$\begin{cases} P_n(x_i,y_0) = f(x_i,y_0) \text{ pentru } i\in\{0,1,...,n\} & (24) \\ P_n(x_i,y_1) = f(x_i,y_1) \text{ pentru } i\in\{0,1,...,n-1\} & (24') \\ \vdots & \vdots \\ P_n(x_i,y_{n-1}) = f(x_i,y_{n-1}) \text{ pentru } i\in\{0,1\} & (24^{n-1}) \\ P_n(x_0,y_n) = f(x_0,y_n) & (24^{n}) \end{cases}$$

Expanding the notes for the divided differences (5) of the function $f(x,y)$ with $(x,y)\in D$, then we have:

$$\begin{cases} f[x_i;y_j] = f(x_i,y_j); \\ f[x_0,x_1;y_j] = \dfrac{f[x_1,y_j]-f[x_0,y_j]}{x_1-x_0}; \\ f[x_i;y_j,y_{j+1}] = \dfrac{f[x_i,y_{j+1}]-f[x_i,y_j]}{y_{j+1}-y_j}; \\ \vdots \\ f[x_0,x_1,...,x_i;y_0,y_1,...,y_j] = \dfrac{f[x_0,x_1,...,x_i;y_1,...y_j]-f[x_0,x_1,..x_i;y_0,y_1,...,y_{j-1}]}{y_j-y_0} = \\ = \dfrac{f[x_1,x_2,...,x_i;y_0,y_1,...,y_j]-f[x_0,x_1,...,x_{i-1};y_0,y_1,...,y_j]}{x_i-x_0} \\ \text{for } i,j\in\{0,1,...,n\} \textit{ and } i+j\le n \end{cases} \tag{25}$$

Considering the conditions (24) on the polynomial (23), we find:

$$\begin{aligned} &a_{i,0} = f[x_0, x_1, ..., x_i; y_0] \text{ pentru i} \in \{0,1,...,n\} \\ &a_{i,1} = f[x_0, x_1, ..., x_i; y_0, y_1] \text{ pentru } i \in \{0,1,...,n-1\} \\ &\vdots \\ &a_{i,n-1} = f[x_0, x_1, ..., x_i; y_0, y_1, ..., y_{n-1}] \text{ pentru } i \in \{0,1\} \\ &a_{0,n} = f[x_0; y_0, y_1, ..., y_n] \end{aligned},$$

which represent the coefficients of the polynomial $P_n(x, y)$, which approximates the function $f(x, y)$ with $(x, y) \in D$.

The error that we make by replacing the function *f* by polynomial of interpolation (23) is given by the relations (20) and (21).

We should notice that the points of divisions Δ for which we know the value of the function f have the characteristic that they can't be found on a curve line (C_n) of the degree *n*, for which the system given by the relations(24),(24'),…,(24^n) be undetermined.

Conclusions

In this paper there were presented only two general cases of interpolation of a function of two variables.

The particular cases when the interpolation polynomial is a linear, bilinear or cubic function by subclasses must also be studied.

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Interpolarea Funcţiilor cu Două Variabile cu Noduri Simple

Rezumat

Problema interpolării unei funcţii de două variabile este o problemă complexă care prezintă dificultăţi majore în rezolvare în funcţie de forma, gradul polinomului de interpolare şi de numărul punctelor din domeniul de definiţie. Fiecare problemă în parte are o metodă de rezolvare care poate constitui o temă de cercetare. În lucrare sunt prezentate două cazuri generale de interpolare cu polinoame de grad n.