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Taking Tomograph by Means of Elastic Wave Non – Homogeneity Shape

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Abstract

The methods of emphasizing the material non-homogeneities of the material analyzed up to now, give directives about their occurring, without being able to provide clear information regarding their feature. Further a detection method is presented, called tomography taking, through which the fault geometry is emphasized.

Key words: *tomography, fault geometry, Fourier transformata*

The Non – Homogeneous Equation of the Wave

The faults represent non homogeneities of the basic medium inducing the *K* fluctuations proportional's its basic value k_0. In this way:

$$K(\vec{r}) = K_0 n(\vec{r}), \tag{1}$$

where $n(r) = c_0 / c(r)$ is the refraction index. In the acoustic approximation, the propagation speed $c(r)$ has the expression:

$$c(\vec{r}) = \frac{1}{\sqrt{\rho(\vec{r})K(\vec{r})}}, \tag{2}$$

where $\rho(r)$ is the density and *K(r)* the compressibility in the medium point with the position vector *r*.

Substituting (1) in atemporal equation, the following wave equation may be obtained:

$$\left(\nabla^2 + K_0^2\right)u(\vec{r}) = -K_0^2\left[n^2(\vec{r}) - 1\right]u(\vec{r}), \tag{3}$$

which may be written in the following shape:

$$\left(\nabla^2 + K_0^2\right)u(\vec{r}) = -f(\vec{r})u(\vec{r}), \tag{4}$$

where $f(\vec{r}) = f(x, y) = -K_0^2\left[n^2(\vec{r}) - 1\right]u(\vec{r})$.

In the case of non homogeneities, the field $u(\vec{r})$ is considered as resulting from summation of two compounds:

$$u(\vec{r}) = u_0(\vec{r}) + u_s(\vec{r}). \tag{5}$$

The quantity $u_0(\vec{r})$ called as incidental field, is the field existing then when the non – homogeneities are absent it being the equation solution: $(\nabla^2 + K_0^2)u_0(\vec{r}) = 0$. $u_s(\vec{r})$ is a part from the total field due to non – homogeneities. Substituting (5) in (4) the following equation is obtained:

$$(\nabla^2 + K_0^2)u_s(\vec{r}) = -f(\vec{r})u(\vec{r}). \tag{6}$$

For obtained $u_s(\vec{r})$ the Green's functions are used, these being solutions for differential equation:

$$(\nabla^2 + K_0^2)g(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}'). \tag{7}$$

$\vec{r}$ and $\vec{r}^{\,'}$ are position vectors of the destination points and a source for Green's function calculation. For a tridimensional infinite space $g(\vec{r}, \vec{r}')$ it has the shape:

$$g(\vec{r}, \vec{r}') = \frac{e^{jk_0|\vec{r}-\vec{r}^{\,'}|}}{4\pi\,|\vec{r} - \vec{r}^{\,'}|}. \tag{8}$$

In the bidimensional case the solution for (7) is a Hankel's function of zero order and the first type:

$$g(\vec{r}, \vec{r}') = \frac{j}{4} H_0^l(K_0\,|\vec{r} - \vec{r}'|). \tag{9}$$

Starting from the Dirac distribution definition: "A Dirac distribution is called a functional given to a function $h(\vec{r})$, continuing in $\vec{r} = 0$, the value $h(0) = \int_{-\infty}^{+\infty} h(\vec{r})\delta(\vec{r})d\vec{r}$ ", the property may be proved:

$$h(\vec{r}) = \int_{-\infty}^{+\infty} h(\vec{r})\delta(\vec{r})d\vec{r}'. \tag{10}$$

Taking into account the fact than $\delta(\vec{r})$ is an even function substituting $h(\vec{r})$ with $f(\vec{r})u(\vec{r})$, the right term of the equation (6) takes the shape:

$$f(\vec{r})u(\vec{r}) = \int f(\vec{r}')u(\vec{r}')\delta\,|\vec{r} - \vec{r}'|\,d\vec{r}'. \tag{11}$$

By using (6), (7), (8) it results that $u_s(\vec{r})$ may be written as a Green's ponderate function:

$$u_s(\vec{r}) = \int f(\vec{r}')u(\vec{r}')g(\vec{r}, \vec{r}')d\vec{r}' \quad \text{or}$$

$$u_s(\vec{r}) = \int f(\vec{r}')u_0(\vec{r}')g(\vec{r}, \vec{r}')d\vec{r}' + \int f(\vec{r}')u_s(\vec{r}')g(\vec{r}, \vec{r}')d\vec{r}'. \tag{12}$$

If it is supposed that $u_s(\vec{r})$ is small in comperis on to $u_0(\vec{r})$, that means that the material non – homogeneity has a wake force of the elastic wave spreading, then the factor $\int f(\vec{r}')u_s(\vec{r}')g(\vec{r}, \vec{r}')d\vec{r}$ may be neglected (Born approximation) and $u_s(\vec{r})$ becomes:

$$u_s(\vec{r}) \approx \int f(\vec{r}')u_0(\vec{r}')g(\vec{r},\vec{r}')d\vec{r}' . \tag{13}$$

In order to solve the equation (13), the Hankel function decomposition from (9) in plane waves will be used:

$$H_0^{(l)}(K_0 \mid \vec{r}-\vec{r}'|) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\beta}e^{j[\alpha|x-x'|+\beta|y-y'|]}d\alpha , \tag{14}$$

where $r=(x,y)$, $r'=(x',y)$ and $\beta=\sqrt{K_0^2-\alpha^2}$. The above mentioned expression, shows effectively a cylindrical wave, H_0 as a superposition of plane waves substituting (14) in (9) and further in (13) the following mode of expression of $u_s(r)$ will be found:

$$u_s(\vec{r}) = \frac{j}{4\pi}\int f(\vec{r}')u_0(\vec{r}')\int_{-\infty}^{+\infty}\frac{1}{\beta}e^{j[\alpha|x-x'|+\beta|y-y'|]}d\alpha d\vec{r}' . \tag{15}$$

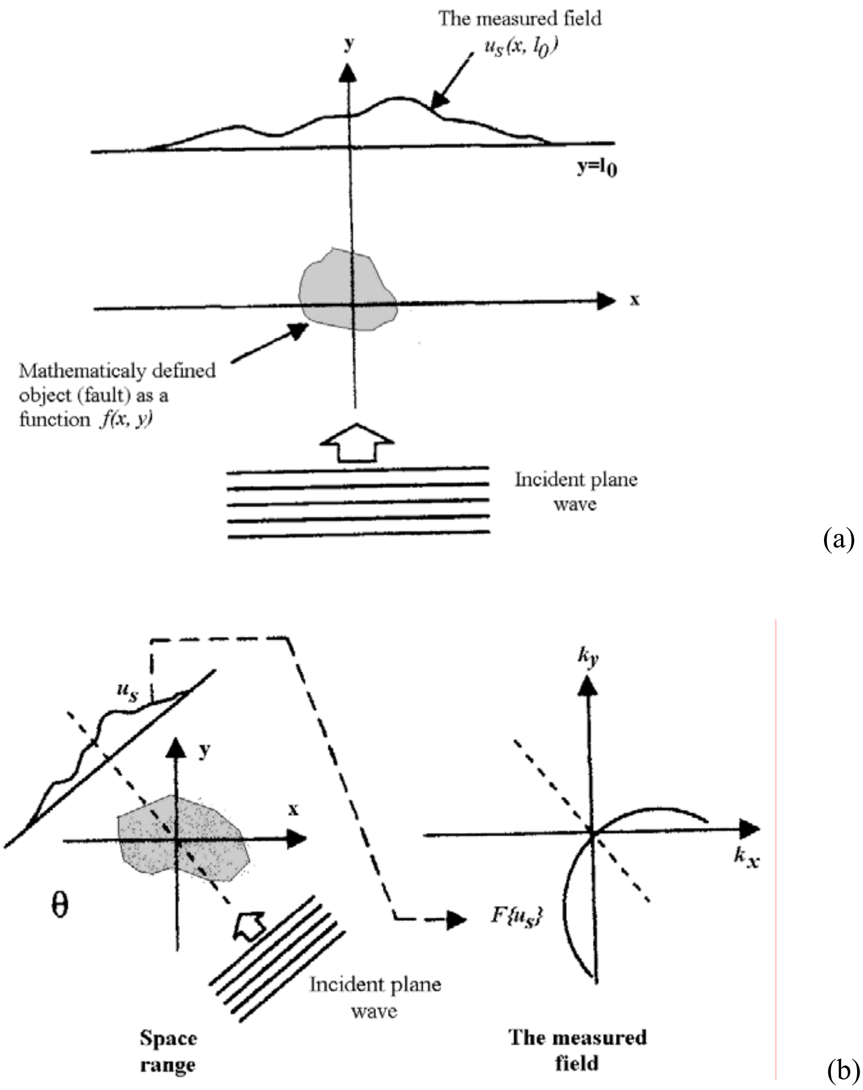

Fig. 1. (a) – Mathematically defined object (fault) as a function $f(x,y)$;
(b) – The 2D Fourier's transformata of the object on a semicircular arc

Further, a coordinate system will be used, in the way that the incident plane wave has a positive direction of the axe $y(u_0(\vec{r}) = u_0(y) = e^{ik_0 y}$ and the straight line on which u_s is measured should be described by equation $y = l_0$, having $l_0 >$ as any y coordinate of the object (as in Fig. 1(a)). As a result, $|y - y'|$ is directly substituted by $|l_0 - y'|$ and (15) becomes:

$$u_s(x, y = l_0) = \frac{j}{4\pi}\int_{-\infty}^{+\infty} d\alpha \int \frac{f(\vec{r}\,')}{\beta} e^{j[\alpha|x-x'|+\beta|l_0-y'|]} e^{jk_0 y'} d\vec{r}\,', \tag{16}$$

$$u_s(x, y = l_0) = \frac{j}{4\pi}\int_{-\infty}^{+\infty} \frac{1}{\beta} e^{j[\alpha x+\beta l_0|} \int f(\vec{r}\,') e^{-j[\alpha x'+(\beta-k_0)y']} d\vec{r}\,'.$$

Being clear the second integral of (16), the bidimensional Fourier's transformata of the function representing the object, evaluated for $k_{x,} k_y = (\alpha, \beta - k_0)$, u_s it is written as follows:

$$u_s(x, y = l_0) = \frac{j}{4\pi}\int_{-\infty}^{+\infty} \frac{1}{\beta} e^{j(\alpha x+\beta l_0)} \left[F_{2D}\{f(\vec{r})\}(\alpha\beta - k_0)\right] d\alpha. \tag{17}$$

On the other hand the monodimensional Fourier's transformata of $u_s(x, l_0)$, (Fig.1) reported to x has the expression:

$$F\{u_s(x, l_{0)}\}(k_s) = \int_{-\infty}^{+\infty} u_s(x, l_0) e^{-jk_s x} dx. \tag{18}$$

Substituting (17) in (18) will result:

$$F\{u_s(x, l_{0)}\}(k_s) = \int_{-\infty}^{+\infty} \frac{j}{4\pi}\int_{-\infty}^{+\infty} \frac{1}{\beta} e^{j(2x+\beta l_0)} \left[F_{2D}\{f(\vec{r})\}(\alpha_1\beta - k_0)\right] e^{-jk_0 x} d\alpha dx. \tag{19}$$

By using the following property of the Fourier's integral:

$$F\{e^{j\alpha k}\} = \int_{-\infty}^{+\infty} e^{j\alpha x} e^{-jk_s x} dx = 2\pi\delta(k_s - \alpha). \tag{20}$$

Where α plays the role of ω_0 and k_s represents ω, the relation ship may be written as:

$$F\{u_s(x, l_{0)}\}(k_s) = \int_{-\infty}^{+\infty} \left(\frac{j}{2\beta} e^{j\beta l_0} \left[F_{2D}\{f(\vec{r})(\alpha_1\beta - k_0)\}\right]\right) \delta(k_s - \alpha) d\alpha. \tag{21}$$

By using the property (10) of the function δ, the expression (21) is transformed in:

$$F\{u_s(x, l_{0)}\}(\alpha) = \frac{j}{2\sqrt{k_0^2 - \alpha^2}} e^{j\sqrt{k_0^2-\alpha^2 l_0}} \left[F_{2D}\{f(\vec{r})\}(\alpha, \sqrt{k_0^2 - \alpha^2} - k_0)\right] \text{ with } |\alpha| < k_0. \tag{22}$$

The relationship (22) represent the Fourier's Diffraction Theorem, linking the bidimensional Fourier's transformata of the object (of the material non – homogeneity) to the monodimensional Fourier's transformata of the field disturbed by fault, ob the measure line. In the extent that a becomes variable from - k_0 to k_0, the coordinates $(a, \sqrt{k_0^2 - a_2} - k_0$ describe a semicircular arch in the frequency space (k_x, k_y) corresponding to the object whose terminal points are found at the distance $k_0\sqrt{2}$ from the origine (Fig. 1(b)).

The calculation algorithm of the function object $f(x, y)$ from projection may be resumed as follows:

1. it is seized $f(x, y) = 0$, whatever would be x, y;
2. the projection $u_s(x, l_0)$ is measured;
3. the rapide Fourier's transformata of the projection $u_s(x, l_0)$ is made up;
4. the IFFT is divided by the factor $\frac{j}{2\sqrt{k_0^2 - \alpha^2}} e^{j\sqrt{k_0^2 - \alpha^2 l_0}}$, obtaining the 2d Fourier's transformata of the object on a semicircular are as in Fig. 1(b) is presented;
5. the steps 2, 3, 4 are repeated rotation the coordinate system in Fig. 1(a), against its centre with the angles $\theta_1, \theta_2, \theta_3, ...\theta_n \in [0,...360^o]$, the projection measuring being made at a distance of l_0 from its origin. In this way, the bidimensional Fourier's transformata of the fault becomes well-know along of a great number of arcs of circle which finally cover a disk. However, on cannot be obtained equally outdistanced samples on the axes k_x and k_y, so that IFFT 2D could be applied;
6. an interpolation algorithm of the obtained samples at the point 5 on a rectangular grid is used;
7. the IFFT 2D of the samples at the point 6 is applied.

Interpolation in the Frequency Range

In the previous paragraph a calculation algorithm of the material non – homogeneity shape when its projections are known had been establish. Nevertheless, the step e) was make sufficiently detailed. Its debating in a distinct paragraph was chosen, because this constituted itself a whole algorithm.

The projection field measuring presented in Fig. 1 (a), in the great majority of cases in carried out by means of a device sameplaying the field $u_s(x, l_0)$ at equidistance space ranges. Therefore, FFT $\{u_s(x, l_0)\}(\alpha)$ will be obtained as a range. By using may be ascertained the sample of Fourier's transformata 2D would not presented point at equal distances (on an each of circle).

Conclusions

About the detectability of a non-homogeneity may be generally said that:

- Increases at the same time with the fault sizes;
- The focusing capacity of the exciting transducer has a decisive effect on the detectability, in other words it is inversely – proportional to the transversal size of the ultrasonaur fascicle;
- It is inversely – proportional to the pulse easting produced by an ultrasonaur excitatory. The shorter the impulsive, the better the detectably is;
- Increases at the same time with the contrast increase between the non-homogeneity and the surrounding medium. In this way, a fault made of a soft material characterized by a little product ρv^2 (where ρ is the material density and v is the propagation speed of the elastic waves) if easier perceptible inside a material than a metallic nature non-homogeneity having a great ρv^2;

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Tomografierea cu Unde Elastice
a unor Neomogenităţi de Material

Rezumat

Tomografierea este o modalitate de calculare a formei unui obiect pornindu-se de la proiecţiile generate prin „iluminarea” sa sub diverse unghiuri cu o radiaţie oarecare, ce poate fi de natură electromagnetică sau elastică. În cazul în care radiaţia folosită are o lungime de undă mult mai mică decât dimensiunile corpului studiat, metoda matematică de reconstrucţie pe baza cunoaşterii proiecţiilor e mai simplă, neînregistrându-se fenomene de difracţie. Pentru lungimi de undă mari, cazul undelor mecanice, difracţia trebuie luată în consideraţie în majoritatea situaţiilor.