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Weel Posedness of Fixed Point Problem in Compact Metric Spaces

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Abstract

The notion of well-posednes of a fixed point problem has generated much interest to a several mathematicians, for examples, F.S. De Blassi and J. Myjak (1989), S. Reich and A. J. Zaslavski (2001), B. K. Lahiri and P. Das (2005). Also, in 2003, V. Popa introduced some fixed point theorems for mappings satisfying a new type of implicit relation. The purpose of this paper is to prove for mappings satisfying a new type of implicit relation in a compact metric space, that fixed point problem is well-posed.

Key words: *well posedness, compact metric space,fixed point, impicit relation*

Introduction

The notion of well posedness of a fixed point problem has generated much interest to a several mathematicians, for examples [2, 4, 8].

Definition 1. Let (X, d) be a metric space and $f: (X, d) \to (X, d)$a mapping. The fixed point problem of f is said to be well posed if:

- f has a uniquefixed point x_0 in X;
- for any sequence $\{x_n\} \in X$: $\lim_{n\to\infty} d(x_n, fx_n) = 0$ we have $\lim_{n\to\infty} d(x_n, x_0) = 0$.

The notion of contractive mapping has been introduced by Banach in [1]. In the last thirty years there have appeared different types of generalizations of this concept.

The connection between them has been studied in different papers [3, 5, 9, 12].

Let (X, d) be a metric space and $T: (X, d) \to (X, d)$ be a mapping. In essence, T is a generalized contraction if a inequality of type

$$d(Tx, Ty) \leq f\left(d(x, y), \mathrm{d}(x, Tx), d(y, Ty), d\big(x, Ty, d(y, Tx)\big)\right) \quad (1)$$

holds for $x, y \in X$, where f:$\mathbf{R}^5 \to \mathbf{R}$ satisfies some properties or has a special form. In [6], the author established a class of mappings f: $\mathbf{R}_+^6 \to \mathbf{R}$ such that fulfillment of the inequality of type

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Ty) \leq 0 \quad (2)$$

for $x, y \in X$ ensures the existence and uniqueness of a fixed point for T.

Recently [7], the present author established two classes of mappings F, G: $\mathrm{R}_+^6 \to \mathbf{R}$ such that the fulfillment of the inequality of type

$$F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(y,T_x^2),d(y,Tx)) \leq 0 \quad (3)$$

or $$G(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d\left(x,T_y^2\right),d(x,Ty) \leq 0 \text{ for } x,y \in X \quad (4)$$

ensures the existence and uniqueness of a fixed point for *T*.

The purpose of this paper is to introduced a new class of mappings $F,G:\mathbf{R}_+^6 \to \mathbf{R}$ such that the fulfillment of the inequality of type

$$F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(y,T_x^2),d(Tx,T_x^2)) \leq 0 \text{ for } x,y \in X, \quad (5)$$

ensures the existence and uniqueness of a fixed point of *T* and to prove for mappings *T* satisfying an implicit relation of type (5) in a compact metric space that the fixed point problem is a well-posed.

Implicit Relations

Let $F(t_1,t_2,\ldots,t_6):\mathbf{R}_+^6 \to \mathbf{R}$ be a continuous mapping. We define the following properties:

(F_i): For every $u \geq 0, v > 0, F(u,v,v,u,u,u) < 0$ i implies $u < v$,

(F_u): $F(t,t,0,0,t,0) > 0, \forall t > 0,$

(F_p): There exists $p \in (0,1)$ such that for every u≥0, v> $0, w \geq 0$ with $F(u,v,o,w,v,o) < 0$ we have $u < pmax\{v,w\}$.

Example1. $F(t_1,t_2,\ldots,t_6) = t_1 - a_1t_2 - a_2t_3 - a_3t_4 - a_4t_5 - a_5t_6,$,

where $a_1 > 0, a_2,\ldots,a_5 \geq 0$ and $0 < a_1 + a_2 + \cdots + a_5 < 1$.

(F_i): Let $u \geq 0, v > 0$ and $F(u,v,v,u,u,u) = u - a_1v - a_2v - a_3u - a_4u - a_5u < 0$.

Then $u < \frac{a_1+a_2}{1-(a_3+a_4+a_5)}v < v$.

(F_u): $F(t,t,0,0,t,0) = t(1 - a_1 - a_4) > 0,\ \forall t > 0$.

(F_p) : Let $u \geq o, v > o, w \geq o$ and $F(u,v,o,w,v,o) = u - a_1v - a_3w - a_4v < o$.

Then $u < a_1v + a_3w + a_4v \leq (a_1 + a_3 + a_4)max\{v,w\}$.Therefore, $u < pmax\{v,w\}$,where $0 < p = a_1 + a_3 + a_4 < 1$.

Example 2. $F(t_1,t_2,\ldots,t_6) = t_1 - cmax\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}$, where $0 < c < 1$.

(F_i): Let u≥ $0, \text{v} > 0$ and $F(u,v,v,u,u,u) = u - cmax\{u,v\} < 0$. If $u > 0$ and $u \geq \text{v}$ then $u(1-c) < 0$, a contradiction. Hence $u < v$. If $u = 0$ then $u < v$.

(F_u): $F(t,t,0,0,t,0) = t(1-c) > 0, \forall t > 0$.

(F_p): Let $F(u,v,w,v,0) = u - cmax\{v,w\} < 0$, which implies $u < pmax\{v,w\}$, where $0 < p - c < 1$.

Example 3. $F(t_1,t_2,\ldots,t_6) = t_1^3 - at_1^2t_2 - bt_1t_2^2 - ct_2\ t_3\ t_4 - dt_5^2t_6$ where $a,b,c,d > 0$ and $a + b + c + d < 1$.

(F_i): Let $u > 0, v >$and $F(u,v,v,u,u,u) = u^3 - a^2v - buv^2 - cv^2u - du^3 < 0$, which implies $\mathrm{u}^2(1-\mathrm{d}) - \mathrm{auv} - \mathrm{v}^2(\mathrm{b}+\mathrm{c}) < 0$, then $f(t) = (b+c)t^2 + at - (1-d) < 0$, where $\mathrm{t} = \frac{\mathrm{v}}{\mathrm{u}}$. Since $f(1) = (a+b+c+d) + 1 < 0$, let $r > 1$ be the root of equation $f(t) = 0$. Then $f(t) > 0$ for $t > 0$, which implies $\mathrm{u} < \frac{1}{\mathrm{r}}\mathrm{v} < v$.

If $u = 0$, then $u < v$.

(F_u): $F(t,t,0,0,t,0) = t(1-a-b) > 0, \forall t > 0$.

(F_p): Let $u > 0, v > 0, w > 0$ and $F(u,v,0,w,v,0) = u^3 - au^2v - buv^2 < 0$, which implies $f(t) = bt^2 + at - 1 > 0$, where $t = \frac{v}{u}$. Since $f(1) = a + b - 1 < 0$, let $r \geq 1$ be the root of equation $f(t) = 0$. Then $f(t) > 0$f for $t > r$, which implies $u \leq pv \leq p\,max\{v,w\}$, where $p = \frac{1}{r} < 1$. If $u = 0$ then $u < p \max\{v,w\}$.

Exemple 4. $F(t_1,t_2,\dots,t_6) = t_1^2 - at_2^2 - \frac{bt_5t_6}{1+t_3+t_4}$, where $a,b > 0$ and $a + b < 1$.

(F_i): Let $u \geq 0, v > 0$ and $F(u,v,v,u,u,u) = u^2 - av^2 - b\frac{u^2}{1+u+v} < 0$.

Then $u^2 - av^2 - bu^2 < 0$, which implies $u < \frac{a}{1-b}v < v$.

(F_u): $F(t,t,0,0,t,0) = t^2(1-a) > 0, \forall t > 0$.

(F_p): Let $u \geq 0,\ v > 0,\ w \geq 0$ and $F(u,v,0,w,v,0) = u^2 - av^2 < 0$, which implies $u < \sqrt{a}\, v \leq p \max\{v,w\}$, where $p = \sqrt{a} < 1$.

Example 5. $F(t_1,t_2,\dots,t_6) = t_1^2 - t_1\,(at_2 + bt_3 + ct_4) - dt_5\,t_6$, where $a > 0, b, c, d \geq 0$ and $a + b + c + d < 1$.

(F_i): Let $u > 0, v > 0$ and $F(u,v,v,u,u,u) = u^2 - u(av + bv + cu) - dv^2 < 0$. Then $u < \frac{a+b}{1-c-d} < v$. If $u = 0$, then $u < v$.

(F_u): $F(t,t,0,0,t,0) = t^2(1-a) > 0, \forall t > 0$.

(F_p): Let $u > 0, v > 0, w \geq 0$ and $F(u,v,0,w,v,0) = u^2 - u(av + cw) < 0$.

Then $u < av + cw \leq (a+c)\max\{v,w\}$. Hence $u < pmax\{v,w\}$, where $0 < p = a + c < 1$. If $u = 0$, then $u < p\,max\{v,w\}$.

Main Results

Theorem 1. Let (X,d) be a metric space and $T:(X,d) \to (X,d)$be a mapping satisfying inequality (5) for every $x \neq y$, where F satisfies the condition (F_u). Then T has at most one fixed point.

Proof. Suppose that T have two fixed points, u and v with $u \neq v$. Then, by inequality (5) we have

$$F\big(d(Tu,Tv), d(u,v), d(u,Tu), d(v,Tv), d(v,T^2u), d(Tu,T^2u)\big) < 0,$$

$$F(d(u,v), d(u,v), 0, 0, d(u,v), 0) < 0.$$

A contradiction of (F_u).

Theorem 2. Let T be a continuous mapping of the compact metric space (X,d) into itself such that the inequality (5) holds for every $x \neq y$, where F satisfies the condition (F_i) and (F_u). Then T has a unique fixed point.

Proof. Let $f(x) = d(x,Tx)$ for all $x \in X$. Since T is continuous, f is continuous. There exists a point $z \in Z$ such that $f(z) = \inf\{Tx : x \in X\}$. Suppose that $z \neq Tz$. Then by inequality (5) for $x = z$ and $y = Tz$ we obtain

$$F\big(d(Tz,T^2z), d(z,Tz), d(z,Tz), d(Tz,T^2z), d(Tz,T^2z), d(Tz,T^2z)\big) < 0,$$

which implies by (F_i) that $d(Tz,T^2z) < d(Tz,z) = \inf\{d(x,Tx) : x \in X\}$, a contradiction. Hence $z = Tz$. By Theorem 1, z is the unique fixed point of T.

Remark 1. From examples 1-5 we obtain five fixed point theorems.

Theorem 3. Let T be a continuous mapping of the compact metric space (X,d) into itself such that the inequality (5) holds for every $x \neq y$, where T satisfies the properties (F_i), (F_u) and (F_p). Then the fixed point problem of T is well-posed.

Proof. By Theorem 2, T has a unique fixed point x_o, i.e. $x_o = Tx_o$. Let $\{x_n\}$ be a sequence in X such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Then by inequality (3.1) we have successively

$$F\big(d(Tx_0, Tx_n)d(x_0, x_n), d(x_0, Tx_0), d(x_n, Tx_n), d(x_n, T^2x_0), d(Tx_0, T^2x_0)\big) < 0,$$

$$F(d(x_0, Tx_n), d(x_0, x_n), 0, d(x_n, Tx_n), d(x_n, x_0), 0) < 0.$$

By (F_p) we have $d(x_0, Tx_n) < p\max\{d(x_0, x_n), d(x_n, Tx_n)\} \leq p\big(d(x_0, x_n) + d(x_n, Tx_n)\big)$.

Therefore $d(x_0, x_n) \leq d(x_0, Tx_n) + d(Tx_n, x_n) < \mathrm{p}(d(x_0, x_n) + d(x_n, Tx_n)) + d(x_n, Tx_n)$.

Which implies $d(x_0, x_n) < \frac{1+\mathrm{p}}{1-\mathrm{p}} d(x_n, Tx_n) \to 0$ if $n \to \infty$. This proves the theorem.

References

1. Banach, S. - Sur les operations dans les ensembles abstraits et leur applications aux équations integrales, *Fundamenta Mathematica* (3), pp. 133-181, 1922
2. De Blassi, F.S., Myjak, J. - Sur la porosite des contractions sans point fixe, *Comptes Rendus Académie Sciences Paris*, (308), pp. 51-54, 1989
3. Kincses, J., Totik, V. - Theorems and counter-examples on contractive mappings, *Mathematica Balcanca*, 4, pp. 69-90, 1990
4. Lahiri, B.K., Das, P. – Well-posednes and porosity of certain classes of operators*,* *Demonstratio Mathematica* (38), pp. 170-176, 2005
5. Park, S. – On general contractive type contractions, *Journal Korean Mathematics Society,* 17, pp. 131-140, 1980
6. Popa, V. – Fixed point theorems for implicit contractive mappings, *Studii şi Cercetări Ştiinţifice, Seria Matematică, Universitatea Bacău,* (7), pp. 129-133, 1997
7. Popa, V. – On some fixed point theorems for mappings satisfying a new type of implicit relation, *Mathematica Moravica,* (7), pp. 61-66, 2003
8. Reich, S., Zaslavski, A.J. – Well-posednes of fixed point problems, *Far East Journal Mathematical Sciences*, Special volume 2001, Part III, pp. 393-401, 2001
9. Rhoades, B.E. – A comparision of various definitions of contractive mappings, *Translation American Mathematical Society,* 26, pp. 255-290, 1971
10. Rhoades, B.E. – A collection of contractiv definitions, *Mathematical Seminar Notes,* 7, pp. 229-235, 1979
11. Rhoades, B.E. – Contractive definitions, Non Linear Analysis Ed. T. M. Rassias, *World Scientific Publishing Company*, New Jersey, pp. 513-526, 1988
12. Tasković, M.R. – Some new principles in fixed point theory, *Mathematica Japonica*, 35, pp. 645-666, 1990

O bună punere a problemei de punct fix în spaţii metrice compacte

Rezumat

Noţiunea de bună-punere a problemei de punct fix a fost studiată cu mult interes de mai mulţi matematicieni, de exemplu F. S. De Blassi şi J. Myjak (în 1989), S. Reich şi J. Zalavski (în 2001), B. K. Lahiri şi P. Das (în 2005). De asemenea, în 2003 V. Popa a introdus un nou tip de relaţie implicită. Scopul lucrării de faţă este de a demonstra pentru aplicaţii ce satisfac un nou tip de relaţie implicită că problema de punct fix este bine-pusă.