BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LX No. 1/2008	11 - 18	Seria Matematică - Informatică - Fizică

On the Topology of Gelfand-Shilov-Roumieu Spaces

Mihai Pascu

Universitatea Petrol-Gaze din Ploieşti, Bd. Bucureşti 39, Ploieşti, Catedra de Matematică
Institutul de Matematică al Academiei Române, Calea Griviţei, 21, Bucureşti
e-mail: Mihai.Pascu@imar.ro

Abstract

We gather here seven definitions of a Gelfand-Shilov-Roumieu space of rapidly decreasing functions as an inductive limit of a directed family of Banach spaces. We prove that all the inductive limit topologies defined are the same: they do not depend on the directed family we use for the definition of the Gelfand-Shilov-Roumieu space.

Key words: *Gelfand –Shilov-Roumieu space, inductive limit topological vector space*

Introduction

The Schwartz space $\mathcal{S}$ of rapidly decreasing functions is the space of smooth functions $\varphi : \boldsymbol{R}^n \to \boldsymbol{C}$ who have the property that

$$\sup_{x\in \boldsymbol{R}^n} \left| x^\beta \partial^\alpha \varphi(x) \right| < \infty,\ (\forall)\alpha, \beta \in \boldsymbol{N}^n .$$

Relatively late, in 1993 (Schwartz published his celebrated books on distribution theory in the fifties), J. Chung, S. Y. Chung and D. Kim ([1]) proved that a function φ belongs to the space $\mathcal{S}$ if and only if

$$\sup_{x\in \boldsymbol{R}^n} \left| x^\beta \varphi(x) \right| < \infty,\ (\forall)\beta \in \boldsymbol{N}^n \ \text{ and } \ \sup_{x\in \boldsymbol{R}^n} \left| \partial^\alpha \varphi(x) \right| < \infty,\ (\forall)\alpha \in \boldsymbol{N}^n ,$$

or, equivalently, if and only if

$$\sup_{x\in \boldsymbol{R}^n} \left| x^\beta \varphi(x) \right| < \infty,\ (\forall)\beta \in \boldsymbol{N}^n \ \text{ and } \ \sup_{x\in \boldsymbol{R}^n} \left| \xi^\alpha \hat{\varphi}(\xi) \right| < \infty,\ (\forall)\alpha \in \boldsymbol{N}^n .$$

Here $\hat{\varphi}$ denotes, as usually, the Fourier transform of the function φ. Notice that in the last characterization of the space of rapidly decreasing functions, the derivatives of the functions are not explicitly involved.

If one imposes more precise conditions on the rate of decay of the function and of its derivatives (or of its Fourier transform) one obtains the Gelfand-Shilov-Roumieu (GSR) spaces. The first definition (chronologically and the most used) of these spaces is the following. If $(M_p)_p$ and $(N_p)_p$ are two sequences of positive numbers who satisfy some additional conditions (the sequences we are working with in our paper will satisfy conditions (A1) – (A4) from below), the GSR space $\mathcal{S}(\{M_p\},\{N_p\})$ is the space of those rapidly decreasing functions who have the property that

$$\sup_{x\in\mathbf{R}^n}\left|x^\beta\partial^\alpha\varphi(x)\right| < CA^{|\alpha|}B^{|\beta|}M_{|\alpha|}N_{|\beta|},\ (\forall)\alpha,\beta\in\mathbf{N}^n$$

for some positive constants *A*, *B* and *C* ([3], [7]).

The conditions (A1) – (A4) are the following:

(A1) $M_0 = 1,\ M_1 \geq 1$;

(A2) $M_p^2 \leq M_{p-1}M_{p+1},\ (\forall)p \geq 1$ (logarithmic convexity);

(A3) there exists a constant $H_1 \geq 1$ such that

$$M_{p+q} \leq H_1^{p+q}M_pM_q,\ (\forall)p,q \geq 0$$

(the condition of stability under ultradifferential operators ([6]));

(A4) there exists a constant $H_2 \geq 1$ such that

$$\sqrt{p}M_{p-1} \leq H_2M_p,\ (\forall)p \geq 1.$$

Starting from this point we shall consider, in order to simplify the notations and the proofs, that $n = 1$. Generalizations of the definitions and results from below to the case $n > 1$ are straightforward.

In order to introduce a topology on $\mathcal{S}(\{M_p\},\{N_p\})$ one defines, for some fixed constants *A* and *B* the norms

$$\|\varphi\|_{A,B}^{(1)} = \sup_{|\alpha|,|\beta|\geq 0}\sup_{x\in\mathbf{R}^n} A^{-|\alpha|}B^{-|\beta|}(M_{|\alpha|}N_{|\beta|})^{-1}\left|x^\beta\partial^\alpha\varphi(x)\right|.$$

If we define

$$\mathcal{S}_{A,B}^{(1)}(\{M_p\},\{N_p\}) = \{\varphi\in\mathcal{S};\ \|\varphi\|_{A,B}^{(1)} < \infty\},$$

then $\mathcal{S}_{A,B}^{(1)}(\{M_p\},\{N_p\})$ is a Banach space and

$$\mathcal{S}(\{M_p\},\{N_p\}) = \bigcup_{A,B>0}\mathcal{S}_{A,B}^{(1)}(\{M_p\},\{N_p\}).$$

We can define therefore an inductive limit topology on $\mathcal{S}(\{M_p\},\{N_p\})$:

$$\mathcal{S}(\{M_p\},\{N_p\}) = \liminf_{A,B>0}\mathcal{S}_{A,B}^{(1)}(\{M_p\},\{N_p\}).$$

The inductive limit is not strict ([7]).

In analogy with the case of the space $\mathcal{S}$ who admits different characterizations, there are also different characterizations of the same GSR space. Correspondingly, we can define different families of norms on subspaces of a GSR space and construct the topology on $\mathcal{S}(\{M_p\},\{N_p\})$ as the inductive limit of different families of Banach spaces. Depending on the purpose we have in mind, it is convenient to use one or another of these characterizations. For example, the last characterization we shall give in the next section appeared to us to be extremely useful when doing the time frequency analysis of GSR spaces and of their duals, the spaces of ultradistributions. So, it is important to know that the topology of the GSR space is the same.

In our paper we shall give 7 equivalent definitions of a GSR space as a set (already known in fact in the mathematical literature ([2], [4], [5], [7]) and we shall prove that the topologies defined starting from these definitions are the same. Even if tacitly assumed in some of the paper mentioned above, we could not find a clear statement on the equality of these topologies and a complete proof of their equality. When the reference is clear we shall indicate it and we shall omit the details.

In the second section of the paper we shall give the basic definitions and we shall formulate the main result. The third section will contain the proofs.

Main Result

We shall denote the operator of multiplication with the variable x with the same letter, x. The L^2 and L^∞ norms for functions defined on $\boldsymbol{R}$ are denoted with $\| \ \|_2$, respectively $\| \ \|_\infty$.

For φ a smooth function on $\boldsymbol{R}$ and A, B two positive constants, we put:

$$\|\varphi\|_{A,B}^{(2)} = \sup_{p,q\geq 0}\left\|A^{-p}B^{-q}(M_pN_q)^{-1}x^q\varphi^{(p)}\right\|_2,$$

$$\|\varphi\|_{A,B}^{(3)} = \sup_{q\geq 0}\left\|B^{-q}N_q^{-1}x^q\varphi\right\|_\infty + \sup_{p\geq 0}\left\|A^{-p}M_p^{-1}\varphi^{(p)}\right\|_\infty,$$

$$\|\varphi\|_{A,B}^{(4)} = \sup_{q\geq 0}\left\|B^{-q}N_q^{-1}x^q\varphi\right\|_2 + \sup_{p\geq 0}\left\|A^{-p}M_p^{-1}\varphi^{(p)}\right\|_2,$$

$$\|\varphi\|_{A,B}^{(5)} = \sup_{q\geq 0}\left\|B^{-q}N_q^{-1}x^q\varphi\right\|_\infty + \sup_{p\geq 0}\left\|A^{-p}M_p^{-1}\xi^p\hat{\varphi}\right\|_\infty,$$

$$\|\varphi\|_{A,B}^{(6)} = \sup_{q\geq 0}\left\|B^{-q}N_q^{-1}x^q\varphi\right\|_2 + \sup_{p\geq 0}\left\|A^{-p}M_p^{-1}\xi^p\hat{\varphi}\right\|_2.$$

Also, for a, $b > 0$, we put

$$\|\varphi\|_{a,b}^{(7)} = \left\|\varphi\, \mathrm{e}^{N(b|x|)}\right\|_\infty + \left\|\hat{\varphi}\, \mathrm{e}^{M(b|\xi|)}\right\|_\infty.$$

Here $M, N : (0,\infty) \to [1,\infty)$ are the functions asociated to the sequences $(M_p)_p$, respectively $(N_p)_p$:

$$M(r) = \sup_{p\geq 0}(p\ln r - \ln M_p),\ (\forall) r > 0.$$

The spaces

$$S_{A,B}^{(i)}(\{M_p\},\{N_p\}) = \{\varphi \in \mathcal{S};\ \|\varphi\|_{A,B}^{(i)} < \infty\},\ i = 1,\dots,6$$

and

$$S_{a,b}^{(7)}(\{M_p\},\{N_p\}) = \{\varphi \in \mathcal{S};\ \|\varphi\|_{a,b}^{(7)} < \infty\}$$

are Banach spaces and the families

$$(S_{A,B}^{(i)}(\{M_p\},\{N_p\}))_{A,B>0},\ i = 1,\dots,6 \text{ and } (S_{a,b}^{(7)}(\{M_p\},\{N_p\}))_{a,b>0}$$

are directed families of Banach spaces (the last family is a directed one since the functions associated to sequences of numbers are nondecreasing functions). Therefore we can define their inductive limit.

Our main result is the following.

Proposition 1. If sequences $(M_p)_p$ and $(N_p)_p$ satisfy the assumptions (A1) – (A4), then

$$\mathcal{S}(\{M_p\},\{N_p\}) = \bigcup_{A,B>0} \mathcal{S}^{(i)}_{A,B}(\{M_p\},\{N_p\}), \text{ for } i = 1,\dots,6$$

and

$$\mathcal{S}(\{M_p\},\{N_p\}) = \bigcup_{a,b>0} \mathcal{S}^{(7)}_{a,b}(\{M_p\},\{N_p\}).$$

The corresponding inductive limits coincide.

Proofs

The starting point of the proof of Proposition 1 is the following lemma on inductive limit topologies.

Lemma 1. Let $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$ be two directed families of normed spaces so that if $X_{i_1} \subset X_{i_2}$ or $Y_{j_1} \subset Y_{j_2}$ then the inclusion operator is bounded. If for every $i \in I$ there exists some $j \in J$ such that X_i is continuously embedded in Y_j and for every $j \in J$ there exists some $i \in I$ such that Y_j is continuously embedded in X_i, then

$$\liminf_{i\in I} X_i = \liminf_{j\in j} Y_j.$$

Proof. Let us recall that the inductive limit topology is defined by specifying a fundamental system of neighborhoods of the origin. The fundamental system of neighborhoods is constituted by the convex and balanced sets such that $W \cap X_i$ (respectively $W \cap Y_j$) is a neighborhood of the origin in X_i for every $i \in I$ (respectively $W \cap Y_j$ is a neighborhood of the origin in Y_j for every $j \in J$). So it is sufficient to prove that $W \cap X_i$ is a neighborhood of the origin in X_i for every $i \in I$ if and only if $W \cap Y_j$ is a neighborhood of the origin in Y_j for every $j \in J$.

Let us assume that $W \cap X_i$ is a neighborhood of the origin in X_i for every $i \in I$ and let $j \in J$. There exist some $i \in I$ such that Y_j is continuously embedded in X_i. Therefore there exists a positive constant C such that

$$\|\varphi\|_{X_i} \le C\|\varphi\|_{Y_j}, \ (\forall)\varphi \in Y_j.$$

Since $W \cap X_i \in \mathcal{V}(0_{X_i})$, there exists a positive constant ε such that $\varphi \in W$ if $\|\varphi\|_{X_i} < \varepsilon$. But if $\varphi \in Y_j$ and $\|\varphi\|_{Y_j} < \dfrac{\varepsilon}{C}$, then $\|\varphi\|_{X_i} < \varepsilon$ and, consequently, $\varphi \in W$. Hence $W \cap Y_j \in \mathcal{V}(0_{Y_j})$.

From Lemma 1 it follows that the conclusion of Proposition 1 holds if each space of type (i) is continuously embedded in a space of type (j) for every $i, j \in \{1,\dots,7\}$. Since the sequences $(M_p)_p$ and $(N_p)_p$ are fixed, we shall write simply $\mathcal{S}^{(i)}_{A,B}$ instead of $\mathcal{S}^{(i)}_{A,B}(\{M_p\},\{N_p\})$. With the letter C we shall denote constants which do not depend of the function φ, on p or on q. Ocasionally we shall attach indexes to constants C.

Lemma 2. $\mathcal{S}^{(1)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(2)}_{A,BH_1}$ where H_1 is the constant from the assumption (A1).

Lemma 3. $\mathcal{S}^{(2)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(1)}_{AH_1,BH_1}$.

The main ideas of the proofs of Lemmas 2 and 3 are contained in the proof of Proposition 1 from the second chapter from [7]. One uses the Schwartz's inequality and the assumptions (A1) - (A3).

Lemma 4. $\mathcal{S}^{(1)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(3)}_{A,B}$ and $\mathcal{S}^{(2)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(4)}_{A,B}$.

Proof. We have that

$$\|\varphi\|^{(3)}_{A,B} \le 2\|\varphi\|^{(1)}_{A,B},\ (\forall)\varphi \in \mathcal{S}^{(1)}_{A,B} \text{ and } \|\varphi\|^{(4)}_{A,B} \le 2\|\varphi\|^{(2)}_{A,B},\ (\forall)\varphi \in \mathcal{S}^{(1)}_{A,B}.$$

Lemma 5. $\mathcal{S}^{(4)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(2)}_{AH_1\sqrt{2H_2},\,2BH_1}$ for $A^2 \ge 2H_2$ and $B \ge 2$.

Before giving the proof let us notice that since the families of Banach spaces are directed, from Lemma 5 it follows that every space of type (4) can be continuously embedded in a space of type (2).

Proof of Lemma 5. We shall follow [2]. We shall give the complete proof (even if it is a little bit tedious) since the estimates from [2] contain some errors. Assume that $\varphi \in \mathcal{S}$. Then using successively integration by parts (the integration by parts is justified since all our functions belong to $\mathcal{S}$), Leibniz rule of differentiation and Schwartz's inequality, we have

$$A^{-p}B^{-q}(M_pN_q)^{-1}\left(\int\left|x^q\varphi^{(p)}(x)\right|^2\mathrm{d}x\right)^{1/2} =$$

$$= A^{-p}B^{-q}(M_pN_q)^{-1}\left(\int x^{2q}\varphi^{(p)}(x)\overline{\varphi}^{(p)}(x)\mathrm{d}x\right)^{1/2} =$$

$$= A^{-p}B^{-q}(M_pN_q)^{-1}\left(\int (-1)^p(x^{2q}\varphi^{(p)})^{(p)}(x)\overline{\varphi}(x)\mathrm{d}x\right)^{1/2} =$$

$$= A^{-p}B^{-q}(M_pN_q)^{-1}\left((-1)^p\sum_{j\le\min(p,2q)}\binom{p}{j}\binom{2q}{j}j!\int x^{2q-j}\varphi^{(2p-j)}(x)\overline{\varphi}(x)\mathrm{d}x\right)^{1/2} \le$$

$$\le A^{-p}B^{-q}(M_pN_q)^{-1}\left(\sum_{j\le\min(p,2q)}\binom{p}{j}\binom{2q}{j}j!\left\|\varphi^{(2p-j)}\right\|_2\left\|x^{2q-j}\varphi\right\|_2\right)^{1/2} =$$

$$= \left[A^{-2p}B^{-2q}\sum_{j\le\min(p,2q)}\binom{p}{j}\binom{2q}{j}j!\frac{M_{2p-j}}{M_p^2}\frac{N_{2q-j}}{N_q^2}(M_{2p-j})^{-1}\left\|\varphi^{(2p-j)}\right\|_2(N_{2q-j})^{-1}\left\|x^{2q-j}\varphi\right\|_2\right]^{1/2}.$$

But

$$M_kM_l \le M_{k+l},\ (\forall)k,l \ge 0,$$

since the sequence $(M_p)_p$ is logarithmic convex. Therefore

$$M_{2p-j}M_j \le CH_1^{2p-j}M_pM_{p-j}M_j \le CH_1^{2p-j}M_p^2$$

and

$$\frac{M_{2p-j}}{M_p^2} \le C H_1^{2p-j} \frac{1}{M_j}.$$

Taking into account this inequality, (A4) and the restrictions imposed on A and B we finally obtain

$$A^{-p}B^{-q}(M_pN_q)^{-1}\left(\int\left|x^q\varphi^{(p)}(x)\right|^2\mathrm{d}x\right)^{1/2} \le$$

$$\le C\left[A^{-2p}B^{-2q}\sum_{j\le\min(p,2q)}\binom{p}{j}\binom{2q}{j}H_1^{2p-j}H_1^{2q-j}\frac{j!}{M_jN_j}(M_{2p-j})^{-1}\cdot\right.$$

$$\left.\cdot\left\|\varphi^{(2p-j)}\right\|_2(N_{2q-j})^{-1}\left\|x^{2q-j}\varphi\right\|_2\right]^{1/2} \le$$

$$\le C'\left[A^{-2p}B^{-2q}2^p2^{2q}\sum_{j\le\min(p,2q)}2^{-p}\binom{p}{j}2^{-2q}\binom{2q}{j}H_1^{2p-j}H_1^{2q-j}H_2^j\cdot\right.$$

$$\left.\cdot(M_{2p-j})^{-1}\left\|\varphi^{(2p-j)}\right\|_2(N_{2q-j})^{-1}\left\|x^{2q-j}\varphi\right\|_2\right]^{1/2} \le$$

$$\le C'\left[\sum_{j\le\min(p,2q)}2^{-p}\binom{p}{j}2^{-2q}\binom{2q}{j}(\sqrt{2^{-1}H_2^{-1}}AH_1^{-1})^{-(2p-j)}(M_{2p-j})^{-1}\left\|\varphi^{(2p-j)}\right\|_2\cdot\right.$$

$$\left.\cdot(BH_1^{-1}2^{-1})^{-(2q-j)}(N_{2q-j})^{-1}\left\|x^{2q-j}\varphi\right\|_2\right]^{1/2} \le$$

$$\le C'\|\varphi\|^{(4)}_{\sqrt{(2H_2)^{-1}}AH_1^{-1},\ 2^{-1}BH_1^{-1}}.$$

The proof is complete.

Lemma 6. $\mathcal{S}^{(4)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(6)}_{A,B}$ and $\mathcal{S}^{(6)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(4)}_{A,B}$.

This lemma is a direct consequence of Plancherel's theorem.

Lemma 7. $\mathcal{S}^{(3)}_{A,B}$ is continuously embedded in $\mathcal{S}^{(2)}_{AH_1\sqrt{2H_2},\ 2BH_1^2}$ for $A^2 \ge 2H_2$ and $B \ge 2$.

Proof. Assume that $\varphi \in \mathcal{S}$. Then we have

$$A^{-p}B^{-q}(M_pN_q)^{-1}\left(\int\left|x^q\varphi^{(p)}(x)\right|^2\mathrm{d}x\right)^{1/2} = A^{-p}B^{-q}(M_pN_q)^{-1}\cdot$$

$$=\left((-1)^p\sum_{j\le\min(p,2q)}\binom{p}{j}\binom{2q}{j}j!\int(1+x^2)x^{2q-j}\varphi^{(2p-j)}(x)\overline{\varphi}(x)\frac{\mathrm{d}x}{1+x^2}\right)^{1/2} \le$$

$$\le A^{-p}B^{-q}(M_pN_q)^{-1}\left(\sum_{j\le\min(p,2q)}\binom{p}{j}\binom{2q}{j}j!\left\|\varphi^{(2p-j)}\right\|_\infty\left\|(1+x^2)x^{2q-j}\varphi\right\|_\infty\right)^{1/2}\left(\int\frac{\mathrm{d}x}{1+x^2}\right)^{1/2} \le$$

$$\leq C\left[A^{-2p}B^{-2q}\sum_{j\leq\min(p,2q)}\binom{p}{j}\binom{2q}{j}j!\frac{M_{2p-j}}{M_p^2}\frac{N_{2q-j}}{N_q^2}(M_{2p-j})^{-1}\left\|\varphi^{(2p-j)}\right\|_\infty\cdot\right.$$

$$\left.\cdot\left((N_{2q-j})^{-1}\left\|x^{2q-j}\varphi\right\|_\infty+\frac{N_{2q-j+2}}{N_{2q-j}}(N_{2q-j+2})^{-1}\left\|x^{2q-j+2}\varphi\right\|_\infty\right)\right]^{1/2}.$$

Following the argument from the second part of the proof of Lemma 5, we obtain the conclusion. The second index of the space of type (2) is equal with $2BH_1^2$ since we have to estimate $\frac{N_{2q-j+2}}{N_{2q-j}}$ using (A3).

Lemma 8. $\mathcal{S}_{A,B}^{(5)}$ is continuously embedded in $\mathcal{S}_{AH_1,\,BH_1}^{(6)}$.

Proof. For φ in $\mathcal{S}$ we have

$$A^{-p}M_p^{-1}\left(\int\xi^{2p}\left|\hat{\varphi}(\xi)\right|^2\,\mathrm{d}\xi\right)^{1/2}=A^{-p}M_p^{-1}\left(\int(1+\xi^2)\xi^{2p}\left|\hat{\varphi}(\xi)\right|^2\frac{\mathrm{d}\xi}{1+\xi^2}\right)^{1/2}\leq$$

$$\leq CA^{-p}M_p^{-1}\left\|(1+|\xi|)\xi^p\hat{\varphi}\right\|_\infty\leq C'\left\|\varphi\right\|_{AH_1^{-1},B}^{(5)}.$$

Similarly

$$B^{-q}N_q^{-1}\left(\int x^{2p}\left|\varphi(x)\right|^2\,\mathrm{d}x\right)^{1/2}\leq C'\left\|\varphi\right\|_{A,BH_1^{-1}}^{(5)}.$$

Lemma 9. $\mathcal{S}_{A,B}^{(1)}$ is continuously embedded in $\mathcal{S}_{A,\,B}^{(5)}$.

Proof. It is sufficient to estimate $\left\|\xi^p\hat{\varphi}\right\|_\infty$. We have

$$A^{-p}M_p^{-1}\left\|\xi^p\hat{\varphi}\right\|_\infty=A^{-p}M_p^{-1}\left\|\int\mathrm{e}^{-i<x,\xi>}\varphi^{(p)}(x)\,\mathrm{d}x\right\|_\infty\leq A^{-p}M_p^{-1}\int\left|\varphi^{(p)}(x)\right|\mathrm{d}x=$$

$$\leq A^{-p}M_p^{-1}\left\|(1+x^2)\varphi^{(p)}\right\|_\infty\int\frac{\mathrm{d}x}{1+x^2}\leq C\left\|\varphi\right\|_{A,B}^{(1)}.$$

For the proof of the last two lemmas we shall again use an idea from [7].

Lemma 10. $\mathcal{S}_{a,b}^{(7)}$ is continuously embedded in $\mathcal{S}_{a^{-1},\,b^{-1}}^{(5)}$.

Proof. We put $B=b^{-1}$. Then, using the definition of the associated function, we obtain that

$$B^{-q}N_q^{-1}\left\|x^q\varphi\right\|_\infty\leq B^{-q}N_q^{-1}B^qN_q\left\|\varphi\,\mathrm{e}^{N(b|x|)}\right\|_\infty=\left\|\varphi\,\mathrm{e}^{N(b|x|)}\right\|_\infty.$$

The Fourier transform of φ is estimated in an analogous manner.

Lemma 11. $\mathcal{S}_{A,B}^{(5)}$ is continuously embedded in $\mathcal{S}_{\delta A^{-1},\,\delta B^{-1}}^{(7)}$, $(\forall)\delta\in(0,1)$.

Proof. We put $a=A^{-1}$, $b=B^{-1}$. From the definition of the associated function it follows that

$$\mathrm{e}^{N(b|x|)}\leq\sum_{q\geq0}\frac{(b|x|)^q}{N_q}.$$

Therefore

$$\left\|\varphi \mathrm{e}^{N(b|x|)}\right\|_\infty \le \left\|\varphi\sum_{q\ge 0}\frac{(b|x|)^q}{N_q}\right\|_\infty = \left\|\varphi\sum_{q\ge 0}\frac{\left(\frac{b}{\delta}|x|\right)^q\delta^q}{N_q}\right\|_\infty \le \|\varphi\|^{(5)}_{A,\delta b^{-1}}\sum_{q\ge 0}\delta^q = \frac{1}{1-\delta}\|\varphi\|^{(5)}_{A,\delta b^{-1}}.$$

Similarly, one has

$$\left\|\hat{\varphi}\mathrm{e}^{M(a|\xi|)}\right\|_\infty \le \frac{1}{1-\delta}\|\varphi\|^{(5)}_{\delta a^{-1},B}.$$

The conclusion of Proposition 1 is derived from the lemmas 2-11.

References

1. Chung, J., Chung, S.Y., Kim, D. – Une caractérisation de l'espace $\mathcal{S}$ de Schwartz, *Comptes Rendues de l'Académie de science de Paris, Série I*, 316, pp. 23-25, 1993
2. Chung, J., Chung, S.Y., Kim, D. – Characterizations of the Gelfand-Shilov spaces via Fourier transforms, *Proceedings of the American Mathematical Society,* 124, 7, pp. 2101-2108, 1996
3. Gelfand, I.M., Shilov, G.E. - *Generalized functions*, vol. 2, Academic Press, New York-London, 1967
4. Grőchenig, K., Zimmermann, G. – Spaces of test functions via the STFT, *Journal of Function Spaces and Applications*, 2, pp. 25-53, 2004
5. Kashpirovsy, A.I. – Equality of the spaces $\mathcal{S}_\alpha^\beta$ and $\mathcal{S}^\alpha \bigcap \mathcal{S}_\beta$, *Functional Analysis and its Applications*, 14, pp. 60-65, 1978
6. Komatsu, H. - Ultradistributions I, structure theorems and characterization, *Journal of the Faculty of Science, University of Tokyo*, Sect. 1A, 20, pp. 25-105, 1973
7. Roumieu, Ch. - Sur quelques extensions de la notion de distribution, *Annales Sciéntifiques de l'École Normale Supérieure*, 77, pp. 47-121, 1960

Asupra topologiei spațiilor Gelfand-Shilov-Roumieu

Rezumat

Reunim aici şapte definiții ale unui spațiu Gelfand-Shilov-Roumieu de funcții rapid descrescătoare ca limită inductivă a unei familii dirijate de spații Banach. Demonstrăm că topologiile limită inductivă astfel definite sînt aceleaşi: ele nu depind de familia dirijată pe care o folosim pentru definirea spațiului Gelfand-Shilov-Roumieu.