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Some New Characterizations of Upper/Lower Almost Nearly Quasicontinous Multifunctions

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Abstract

The paper [26] introduces Rychlewich's notion of upper/lower almost nearly quasicontinous multifunction as a generalization of upper/lower almost quasicontinous multifunction [25] and upper/lower nearly continuous multifunctions [9]. The purpose of our paper ist o obtain new theorems of characterization for upper/lower almost nearly quasicontinuous multifunctions.

Key words: *multifunctions, upper/lower nearly quasicontinuity*

Introduction

The notion of N-closed set is introduced in [6]. The notion of N-continuous function is introduced in [14] and studied in [19, 23] and other papers. Recently, Ekici [8] introduced and studied upper/lower nearly continuous multifunctions as a generalization of upper/lower continuous multifunctions and N-continuous functions. Also, Ekici introduced the notions of upper/lower almost nearly continuous multifunctions as a generalization of upper/lower almost continuous multifunctions [24] and upper/lower nearly continuous multifunctions.

Quite recently, Rychlewicz [26] introduced the notions of upper/lower almost nearly quasicontinuous multifunctions as a generalization of upper/lower almost quasicontinuous multifunctions [25] and upper/lower nearly continuous multifunctions [9].

In this paper we obtain further characterizations of upper/lower almost nearly quasicontinuous multifunctions.

Preliminaries

Let (X, τ) be a topological space and A a subset of X.

The closure of A and the interior of A are denoted by $C\ell(A)$ and $\mathcal{J}nt(A)$, respectively. The subset A of (X, τ) is said to be regular open (resp. Regular colsed) if $A = \mathcal{J}nt(C\ell(A))$ (resp. $A = C\ell(\mathcal{J}nt(A))$).

Definition 1. The subset A is called N-closed (relative to X) if every cover of A by regular open sets of X has a finite subfamily whose union covers A/6/ A point $x \epsilon X$ is called a δ-cluster point [27] of a subset A if $\mathcal{J}nt(C\ell(U)) \cap A \neq \phi$ for every open U of X containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $C\ell_{\delta}(A)$. If $A = C\ell_{\delta}(A)$ then A is said to be δ-closed [27].

The complement of a δ-closed set is said to be δ-open.

The union of all δ-open set contained in A is called the δ-interior of A and is denoted by $\mathcal{J}nt_{\delta}(A)$.

It is shown in [27] that $C\ell_{\delta}(\cup) = C\ell(\cup)$ for every open set U of X and $C\ell_{\delta}$(B) is closed for every subset B of X.

Definition 2. The subset A of a topological space is said to be semi-open [13] (resp. preopen [15], α-open [17], b-open [14], β-open [1] or semi-preopen [31] if

$$A \subset C\ell(\mathcal{J}nt(A)) \left(resp. A \subset \mathcal{J}nt(C\ell(A)), A \subset \mathcal{J}nt(C\ell(A))\right),$$
$$A \subset (C\ell(A)) \cup C\ell(\mathcal{J}nt(A)), A \subset C\ell\left(\mathcal{J}nt(C\ell(A))\right).$$

The family of all semi-open sets of X is denoted by *SO(X).*

The family of all semi-open sets of X contained $x \epsilon X$ is denoted by *SO(X,x).*

Definition 3. The complement of a semi-open (resp. preopen, α-open, b-open, β-open) set is said to be semi-closed [7] (resp. preclosed [15], α -closed [16], b-closed [4], β -closed [1]).

Definition 4. The intersection of all semi-closed (resp. preclosed, α-closed, b-closed, β-closed) sets of X containing A is called the semi-closure [7] (resp. preclosure[10], α-closure [16], b-closure[4], β-closure[2] of A and is denoted by SCℓ(A) (resp. pCℓ(A), αCℓ(A), bC ℓ(A), βC ℓ(A)).

Definition 5. The union of all semi-open (resp. preopen, α-open, b-open, β-open) sets of X contained in A is called the semi-interior (resp. preinterior, α-nterior, b-interior, β-interior) of A and is denoted by

$$S\mathcal{J}nt(A) \text{ (resp. p}\mathcal{J}nt(A), \alpha\mathcal{J}nt(A), b\mathcal{J}nt(A), \ \beta\mathcal{J}nt(A)).$$

The following lemma is a generalization of Lemma 1 [26].

Lema 1. Let V be any preopen set of X having N-closed complement; then $\mathcal{J}$nt(Cℓ(V)) is a regular open set having N-closed complement.

Proof. Since V have N-closed complement, then $X - V$ is N-closed and X-$\mathcal{J}nt(C\ell(V)) \subset X - V$. Let $D = \{D_i : i \in I\}$ be a regular open cover of X-$\mathcal{J}nt(C\ell(V))$. Then $D \cup \mathcal{J}nt(C\ell(V))$ is a regular open cover of X – V.

Since $X - V$ is N-closed there exists a finite I_0 such that $\{D_i : i \epsilon I_0\} \cup \mathcal{J}nt(C\ell(V))$ is a regular open cover of $X - V \supset X - \mathcal{J}nt(C\ell(V))$

Hence $D' = \{D_i : i \epsilon I_0\}$ is a regular open cover of X- $\mathcal{J}nt(C\ell(V))$, hence $\mathcal{J}nt(C\ell(V))$ have N – closed complement.

The following basis properties of semi-closure and semi-interior are useful in the sequel:

Lema 2. Let A be a subset of a topological space (X,τ).

The following holds for the semi-interior and semi-closure of A:

(1) A is semi-closed if and only if $A = sC\ell(A)$;

(2) A is semi-open if and only if $A = s\mathcal{J}nt(A)$;

(3) $sC\ell(-A) = X - s\mathcal{J}nt(A)$, $s\mathcal{J}nt(X_A) = X - sC\ell(A)$.

Definition 6. A function $f: (X, \tau) \to (Y, \sigma)$ is *N*-continuous at a point $x \in X$ [14] if for each open set V of Y containing f(x) and having N-closed complement, there is an open set U containing x such that $f(U) \subset V$. The function $f: (X, \tau) \to (Y, \sigma)$is N-continuous if it has this property at each point $x \in X$.

Throughout the present paper (X, τ) and (Y, σ) always denote topological spaces and $F: (X, \tau) \to (Y, \sigma)$presents a multivalued function. For a multifunction *F:X→Y*, we shall denote the upper and lower inverse of a set *B* of Y by $F^+(B)$ and $F^-(B)$, respectively, that is $F^+(B) = \{x \in X: F(x) \cap B \neq \phi\}$.

Definition 7. A multifunction $F: (X, \tau) \to (Y, \sigma)$ is said to be:

a. Upper nearly continuous [8] (resp. upper almost nearly continuous [9]) at $x \in X$ if for each open set *V* containing *F(x)* and having N-closed complement, there exists an open set *U* containing *x* such that $F(U) \subset V$(resp. *F(U)*$\subset \mathcal{J}nt\big(C\ell(V)\big) = sC\ell(V)$;

b. Lower nearly continuous [8] (resp. lower almost nearly continuous [9] at $x \in X$ if for each open set *V* which intersects *F(x)* and having *N*-closed complement, there exists on open set *U* containing x such that *F(u)*$\cap V \neq \phi$ (resp. *F(u)* $\cap \mathcal{J}nt(C\ell(V)) \neq \phi$) for every $u \epsilon U$;

c. upper/lower nearly continuous (resp. upper/lower almost nearly continuous) if it has this property at each point $x \in X$.

Definition 8. A multifunction $F: (X, \tau) \to (Y, \sigma)$ is said to be

a. upper almost quasi-continuous [25] (resp. upper almost nearly quasi continous [26] at $x \in X$ if for each open set *U* containing *x* and for each open set *V* (resp. for each open set *V* having *N*-closed complement) containing *F(x),* there exists an nonempty open set *G* of *X* such that *G*⊂U and *F(G)*$\subset \mathcal{J}nt(C\ell(V)$= $sC\ell(V)$;

b. lower almost quasi-continuous [25] (resp. lower almost nearly quasi-continuous) at $x \in X$ if for each open set *V* (resp. for each open set *V* having *N*-closed complement) wich intersects *F(x),* there exists a nonempty open set *G*$\subset U$ and *F(g)* $\cap \mathcal{J}nt(C\ell(V) \neq \phi$ for each *g*$\in G$;

c. upper/lower almost quasi-continuous (resp. upper/lower almost nearly quasi-continuous) if it has this property at each $x \in X$.

Theorem 1 [26]. For a multifunction $F: (X, \tau) \to (Y, \sigma)$ the following properties are equivalent:

(1) F is u.a.n.c.;

(2) For any $x \in X$ and for any regular open set *G* of *Y* having *N*-closed complement such that *F(x)* ⊂*G* and for amy open set *U* containing *F(x),* there exists a nonempty open set *W*U such that *F(W)* $\subset G$;

(3) For any $x \in X$ and for any open set *G* of *Y* having connected complement such that *F(x)* $\subset G$, there exists a semi-open set *U* containing *x* such that *F(U)* $\subset \mathcal{J}nt(C\ell(G))$;

(4) F^+*(G)* is a semi-open set for any regular open set *G* of *Y* having *N*-closed complement;

(5) F^-*(K)* is semi-closed set for any regular closed *N*-closed set *K* of *Y*.

Theorem 2 [26]. For a multifunction $F:(X,\tau)\to(Y,\sigma)$ the following properties are equivalent:

(1) F is l.a.n.c.;

(2) For any $x\in X$ and for any regular open set *G* of *Y* meeting *F(x)* and having *N*-closed complement and for any open set *U* containing *x*, there exists a non-empty open set *W* of *X* such that $W\subset U$ and $F(W)\cap G\neq\phi$;

(3) For any $x\in X$ and for any open set *V* of *Y* having *N*-closed complement such that $F(x)\cap V\neq\phi$, there exists a semi-open set *U* containing *x* such that $F(u)\cap \mathcal{J}nt(C\ell(V))\neq\phi$ for every $u\in U$;

(4) $F^-(G)$ is a semi-open set for any regularly open set *G* of *Y* having *N*-closed complement;

(5) $F^+(K)$ is a semi-closed set for any regularly closed *N*-closed set *K* of *Y*.

By Definitions 6, 7 and Theorems 1 and 2 we have u.a.c.⇒u.a.q.c. ⇒; l.a.c⇒l.a.q.c⇒l.a.n.q.c. u.n.c. ⇒u.a.n.c. ⇒u.a.n.q.c.; l.n.c.⇒l.a.n.c.⇒l.a.n.q.c.

Characterizations

Theorem 3. For a multifunction $F:(X,\tau)\to(Y,\sigma)$ the following properties are equivalent:

(1) F is u.a.n.q.c.;

(2) $F^+(V)\subset s\,\mathcal{J}nt(F^+(sC\ell(V)))=s\mathcal{J}nt(F^+(\mathcal{J}nt(C\ell(V))))$ for every open V having N-closed complement;

(3) $sC\ell(F^-\left(C\ell(\mathcal{J}nt(K))\right))\subset F^-(K)$ for every closed N-closed set K of Y;

(4) $sC\ell(F^-\left(C\ell(\mathcal{J}nt(C\ell(B)))\right))\subset F^-(C\ell(B))$ for every subset B of Y having N-closed closure;

(5) $\mathcal{J}nt(C\ell(F^-\left(C\ell(\mathcal{J}nt(K))\right))\subset F^-(K)$ for every closed N-closed set K of Y;

(6) $F^+(V)\subset C\ell(\mathcal{J}nt(F^+(sC\ell(V))))$ for every open set V of Y having N-closed complement.

Proof. (1)⇒(2). Let *V* be any open set *Y* having *N*-closed complement and $x\in F^+(V)$; then $F(x)\subset V\subset sC\ell(V)$ and hence $x\notin X^-F^+(sC\ell(V))$.

By Lemma 2, $sC\ell(V)$ is a regular open set having *N*-closed complement and Y- $s(V)$ is a regular closed *N*-closed in *Y*.

By Theorem 2(5), $F^-(Y\text{-}\,sC\ell(V))$ is a semi-closed set in *X*. Therefore, we obtain

$$x\in F^+(sC\ell(V))\in SO(X)$$

and hence $x\in s\,\mathcal{J}nt(F^+(sC\ell(V)))$. Consequently $F^+(V)\subset s\,\mathcal{J}nt(F^+(sC\ell(V)))$.

(2) ⇒(3). Let *K* be any closed *N*-closed set of *Y*.

Then Y-K is open having N-closed complement. Then we have

$$X\text{-}F^-(K)=F^+(Y\text{-}K)\subset s\mathcal{J}nt(F^+(sC\ell(Y\text{-}K)))=s\mathcal{J}nt(F^+\mathcal{J}nt(C\ell(Y-K)))=s\,\mathcal{J}nt(Y\text{-}C\ell(\mathcal{J}nt(K)))=s\,\mathcal{J}nt(X\text{-}-F^-\left(C\ell(\mathcal{J}nt(K))\right)=X\text{-}\,sC\ell(F^-\left(C\ell(\mathcal{J}nt(K))\right).$$

Therefore we obtain

$$sC\ell(F^-\left(C\ell(\mathcal{J}nt(K))\right)\subset F^-(K).$$

(3) ⇒(4). This is obvious.

(4) ⇒(5). It follows by Lemma 4.1 from [21] that $\mathcal{J}nt(C\ell(S)) \subset sC\ell(S)$ for every subset *S*. Thus for every closed *N*-closed set $\mathbb{K}$ of *Y*, we have

$$\mathcal{J}nt(C\ell(F^-\left(C\ell(\mathcal{J}nt(K))\right))) \subset sC\ell(F^-\left(C\ell(\mathcal{J}nt(K))\right) = sC\ell(F^-\left(C\ell(\mathcal{J}nt(C\ell(K)))\right) \subset F^-(C\ell(K)) = F^-(K).$$

(5)⇒(6). Let V be any open set of *Y* having *N*-closed complement; then *Y-V* is closed *N*-closed in *Y* and by (5) we have

$$\mathcal{J}nt(C\ell(\mathcal{J}nt(Y\text{-}V))) \subset F^-(Y\text{-}V)=X\text{-}(F^+(V).$$

Moreover, we have

$$\mathcal{J}nt(C\ell(F^-(\mathcal{J}nt(Y\text{-}V)))= \mathcal{J}nt(C\ell(F^-(Y\text{-}\mathcal{J}nt(C\ell(V))))= \mathcal{J}nt(C\ell(X\text{-}(F^+(SC\ell(V))))= X\text{-}C\ell(\mathcal{J}nt(F^+(sC\ell(V)))).$$

Therofore, we obtain $(F^+(V) \subset C\ell(\mathcal{J}nt(F^+(SC\ell(V))))$.

(6)⇒(1). Let *x* be any point of *X* and *V* be any open set having *N*-closed complement such that $F(x) \subset V$.

Then $x\epsilon(F^+(V) \subset C\ell(\mathcal{J}nt(F^+(sC\ell(V))))$. Let *U* be any open set containing *x*. Then $G=U\cup \mathcal{J}nt(F^+(sC\ell(V)))\neq \phi$, hence *G* is an non-empty open set contained in *U* with $F(G) \subset sC\ell(V)= \mathcal{J}nt(C\ell(V))$.

By Theorem 2 (2) F is u.a.n.c.

Theorem 4. For a multifunction $F: (X,\tau) \to (Y,\sigma)$ the following properties are equivalent:

(1) F is l.a.n.q.c.;

(2) $F^-(V) \subset s\,\mathcal{J}nt(F^-(sC\ell(V)))$ for every open set *V* having *N*-closed complement;

(3) $sC\ell(F^+(C\ell(\mathcal{J}nt(K))) \subset F^+(K)$ for every closed *N*-closed set *K* of *Y*;

(4) $sC\ell(F^+(C\ell(\mathcal{J}nt(C\ell(B)))) \subset (F^+(C\ell(B))$ for every subset *B* of *Y* having *N*-closed closure;

(5) $\mathcal{J}nt(C\ell((F^+(C\ell(\mathcal{J}nt(K)))) \subset F^+(K)$ for every closed *N*-closed set *K* of *Y*;

(6) $F^-(V) \subset C\ell(\mathcal{J}nt(F^-(SC\ell(V)))$ for every set *V* having *N*-closed complement.

Proof. The proof is similar to the proof of Theorem 3.

Theorem 5. For a multifunction $F: (X,\tau) \to (Y,\sigma)$ the following properties are equivalent:

(1) F is u.a.n.q.c.;

(2) $sC\ell(F^+(V)) \subset F^-(C\ell(V))$ for every β-open set *V* of *Y* having *N*-closed closure;

(3) $sC\ell(F^-(V)) \subset F^-(C\ell(V))$ for every semi-open set *V* of *Y* having *N*-closed closure;

(4) $F^+(V) \subset s\,\mathcal{J}nt(F^+\mathcal{J}nt(C\ell(V))))$ for every preopen set *V* of *Y* having *N*-closed complement.

Proof. (1)⇒(2). Let *V* be any β-open set of *Y* having *N*-closed closure. It follows by Theorem 2.4 from [3] that $C\ell(V)$ is regular closed. Since F is u.a.n.q.c. and $C\ell(V)$ is regular closed *N*-closed, by Theorem 2(5) it follows that $F^-(C\ell(V))$ is semi-closed, hence $sC\ell(F^-(V))=$

$F^-(C\ell(V))$. Therefore, $sC\ell(F^-(V)) \subset sC\ell(F^-(C\ell(V))= F^-(C\ell(V))$, hence $sC\ell(F^-(V)) \subset F^-(C\ell(V))$.

(2) ⇒(3). The proof is obvious since every semi-open set is β-open.

(3) ⇒(1). Let K be any regular closed *N*-closed set of *Y*. Then *K* is semi-open set having *N*-closed closure and hence $sC\ell(F^-(K)) \subset F^-(C\ell(K))= F^-(K)$.Therefore $sC\ell(F^-(K))= F^-(K)$.

By Lemma 2, $F^{-}(K)$ is semi-closed set and by Theorem 2.(5) F is u.a.n.q.c.

(1)⇒ (4). Let V be any preopen set having N-closed complement.

By Lemma 1, $\mathcal{J}nt(C\ell(V))$ is a regular open set having N-closed complement. Then by Theorem 2 we have $F^{+}(V) \subset F^{+}(\mathcal{J}nt(C\ell(V)))$=s $\mathcal{J}nt(F^{+}(C\ell(V))))$ because $F^{+}(\mathcal{J}nt(C\ell(V))$ is semi-open.

(4)⇒ (1). Let V be any regular open set having N-closed complement. Then V is preopen having N-closed complement and hence $F^{+}(V) \subset$ s $\mathcal{J}nt(F^{+}\left(\mathcal{J}nt(C\ell(V))\right) = s\ \mathcal{J}nt(V)$. Hence $F^{+}(V)$= s $\mathcal{J}nt(F^{+}(V))$ and $F^{+}(V)$ is semi-open. By Theorem 2.1 F is u.a.n.q.c.

Theorem 6. For a multifunction $F: (X, \tau) \to (Y, \sigma)$ the following properties are equivalente:

(1) F is l.a.n.q.c.;

(2) $sC\ell(F^{+}(C\ell(V))) \subset F^{+}(C\ell(V))$ for every β-open set V of Y having N-closed closure;

(3) $sC\ell(F^{+}(C\ell(V)) \subset F^{+}(C\ell(V))$ for every semi-open set having N-closed closure;

(4) $F^{-}(V) \subset$ s $\mathcal{J}nt(F^{-}\left(\mathcal{J}nt(C\ell(V))\right)$ for every preopen set V of Y having N-closed complement.

Proof. The proof is similar to proof of Theorem 3.3.

Lemma 3 [22]. For a subset V of a topological space the following properties hold:

1. $\alpha C\ell(V) = C\ell(V)$ for every β-open set V of Y;
2. p $C\ell(V) = C\ell(V)$ for every semi-open set V of Y.

Corollary 1. For a multifunction $F: (X, \tau) \to (Y, \sigma)$ the following properties holds:

1. F is a u.a.n.q.c.;
2. $(SC\ell(F^{-}(V)) \subset F^{-}(\alpha C\ell(V))$ for every β-open set V of Y having N-closed closure;
3. $SC\ell(F^{-}(V)) \subset F^{-}(pC\ell(V))$ for every semi-open set V having N-closed closure.

Corollary 2. For a multifunction $F: (X, \tau) \to (Y, \sigma)$ the following properties are equivalent:

1. F is l.a.n.q.c.;
2. $sC\ell(F^{+}(V)) \subset F^{+}(C\ell(V))$ for every β-open set V of Y having N-closed closure;
3. $sC\ell(F^{+}(V)) \subset F^{+}(pC\ell(V))$ for every semi-open set V of Y having N-closed closure.

Theorem 7. For a multifunction $F: (X, \tau) \to (Y, \sigma)$ the following properties are equivalent:

(1) F is u.a.n.q.c.;

(2) $sC\ell(F^{-}(C\ell(\mathcal{J}nt(C\ell_{\delta}(B))))) \subset F^{-}(C\ell_{\delta}(B))$ for every subset B of Y with $C\ell_{\delta}(B)$ N-closed;

(3) $sC\ell(F^{-}(C\ell(\mathcal{J}nt(C\ell\ \ (B))))) \subset F^{-}(C\ell_{\delta}(B))$ for every subset B of Y with $C\ell_{\delta}(B)$ N-closed;

Proof.(1) ⇒ (2). Let B be any subset of Y with $C\ell_{\delta}(B)$ N-closed.

By Lemma 2 from [27], $C\ell_{\delta}(B)$ is closed. Since $C\ell_{\delta}(B)$ is closed and N-closed then by Theorem 3, $SC\ell(F^{-}\left(C\ell(\mathcal{J}nt(C\ell(B))\right)) \subset F^{-}(C\ell(B))$.

(2)⇒(3). This is obvious since $C\ell\ \ (B) \subset F^{-}C\ell_{\delta}(B)$.

(3) ⇒(1). Let K be a regular closed N-closed set of Y; then by (3) and Theorem 2 from [11] we have $sC\ell(F^{-}(K))$ = s $C\ell(F^{-}(C\ell(\mathcal{J}nt(K))$= s $C\ell(C\ell(\mathcal{J}nt(C\ell(K)))) \subset F^{-}(C\ell_{\delta}(K)) = F^{-}(K)$.

Hence $F^{-}(K)= sC\ell(F^{-}(K))$. By Lemma 2 $F^{-}(K)$ is semi-closed set.

By Theorem 2 F is u.a.n.q.c..

Theorem 8. For a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ the following properties are equivalent:

(1) F is l.a.n.q.c.;

(2) $sC\ell(F^{+}(C\ell(\mathcal{J}nt(C\ell_{\delta}(B))))) \subset F^{+}(C\ell_{\delta}(B))$ for every subset B of Y with $C\ell_{\delta}(B)$ *N*-closed;

(3) $sC\ell(F^{+}(C\ell(\mathcal{J}nt(C\ell\ \ (B))))) \subset F^{+}(C\ell_{\delta}(B))$ for every subset B of Y with $C\ell_{\delta}(B)$ *N*-closed;

Proof. The proof is similar to the proof of Theorem 7.

Definition 9. The subset *A* of a topological space (X,τ) is said to be:

(1) α-regular [12] if for each a∈*A* and each open set *U* containing a, there exists an open *G* of *X* such that a∈*G*⊂ $C\ell(G)$⊂*U*;

(2) α-paracompact [28] if every *X*-open of *A* has an *X*-open refinement which covers *A* and is locally finite for each point of *X*.

Lemma 4 ([23]). If A is an α-regular α-paracompact subset of a topological space (X,τ) and *U* is an open neighborhood of *A*, then there exists an open set G of *X* such that *A*⊂*G*⊂ $C\ell$(G)⊂U.

For a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ by $C\ell(F)$ [5]: $(X,\tau) \to (Y,\sigma)$ /5/ we denote a multifunction defined as follows:

$$C\ell(F)(x)= C\ell(F(x)) \text{ for each } x\in X.$$

Similary, we denote $sC\ell$(F): $(X,\tau) \to (Y,\sigma)$, p$C\ell$(F): $(X,\tau) \to (Y,\sigma)$, $\alpha C\ell$(F): $(X,\tau) \to (Y,\sigma)$, b$C\ell$(F): $(X,\tau) \to (Y,\sigma)$, $\beta C\ell$(F): $(X,\tau) \to (Y,\sigma)$.

Lemma 5. If F: $(X,\tau) \to (Y,\sigma)$ is a multifunction such that *F(x)* is α-regular and α-paracompact for each *x*∈*X,* then $G^{+}(V) = F^{+}(V)$ for every regular open set *V* of *Y* when *G* denotes $C\ell$(F), s$C\ell(F)$, p $C\ell(F)$, α $C\ell(F)$, b $C\ell(F)$, $\beta C\ell$(F).

Proof. Let V any regular open set of Y and x∈ $G^{+}(V)$.

Then *G(x)*⊂V and *F(x)* ⊂G(x)⊂*V.* We have *x*∈ $F^{+}(V)$ and hence $G^{+}(V) \subset F^{+}(V)$. Conversely, let *x*∈ $F^{+}(V)$. Then we have *F(x)*−⊂*V* and by Lemma 4 there exists an open set *H* of *Y* such that *F(x)*⊂H⊂ $C\ell(H)$ ⊂*V*. Since *G(x)* ⊂ $C\ell(F(x))$, *G(x)*−⊂*V* and hence x∈ $G^{+}(V)$. Thus we obtain $F^{+}(V) \subset G^{+}(V)$. Therefore, $G^{+}(V) = F^{+}(V)$.

Lemma 6. For a multifunction *F*: $(X,\tau) \to (Y,\sigma)$, $G^{-}(V) = F^{-}(V)$ for each regular open set of *Y*, where *G* denotes $C\ell(F)$, s$C\ell(F)$, p $C\ell(F)$, α $C\ell(F)$, b $C\ell(F)$, $\beta C\ell(F)$.

Proof. Let V be any regular open set V of Y and x∈ $G^{-}(V)$.

Then G(x)∩ $V \neq \phi$ and hence F(x) ∩ $V \neq \phi$ since V is open. We have x∈ $F^{-}(V)$ and hence $G^{-}(V) \subset F^{-}(V)$. Conversely, let x∈ $F^{-}(V)$. Then we have $\phi \neq$F(x) ∩ $V \subset$ G(x)∩ V and hence x∈ $G^{+}(V)$. Thus we obtain $F^{-}(G) \subset G^{-}(V)$. Therefore, .$F^{-}(V) = G^{-}(V)$.

Theorem 9. Let $F : (X,\tau) \to (Y,\sigma)$ be a multifunction such that *F(x)* is α-regular and α-paracompact for each point *x*∈*X.*
Then the following properties are equivalent.

(1) F is u.a.n.q.c.;

(2) s$C\ell$(F) is u.a.n.q.c;

(3) p $C\ell$(F) is u.a.n.q.c.;

(4) α $C\ell$(F) is u.a.n.q.c.;

(5) b $C\ell$(F) is u.a.n.q.c.;

(6) $C\ell$(F) is u.a.n.q.c.;

(7) $\beta C\ell$(F) is u.a.n.q.c..

Proof. We set G= $C\ell$(F), s $C\ell$(F), p $C\ell$(F), α $C\ell$(F), b $C\ell$(F), $\beta C\ell$(F).

Assuming that F is u.a.n.q.c.. Let *V* any regular open set of *Y* containing *G(x)* and having *N*-closed complement.

Then Theorem 2 and Lemma 5 demonstrate that $G^+(V)$=$F^+(V) \subset SO(X)$. By Theorem 2 *G* is u.a.n.q.c..

Conversely, supposing that *G* is u.a.n.q.c.. Let *V* any regular open set of *Y* containing *F(x)* and having connected complement. Then it follows by Theorem 2 and Lemma 5 that $F^+(V) = G^+(V)$ ∈*SO(X*). By Theorem 2 *F* is u.a.n.q.c..

Theorem 10. For a multifunction $F : (X,\tau) \to (Y,\sigma)$ the following properties are equivalent:

(1) is l.a.n.q.c.;

(2) $C\ell$(F) is l.a.n.q.c.;

(3) (3) s$C\ell$(F) is l.a.n.q.c;

(4) p $C\ell$(F) is l.a.n.q.c.;

(5) α $C\ell$(F) is l.a.n.q.c.;

(6) b $C\ell$(F) is l.a.n.q.c.;

(7) $\beta C\ell$(F) is l.a.n.q.c..

Proof. The proof is similar to the proof of Theorem 9.

Theorem 11. Let (X,τ) be a topological space and $\{U_i : i \in I\}$ a cover of X by α-open sets of (X,τ). A multifunction $F : (X,\tau) \to (Y,\sigma)$ is u.a.n.q.c. if and if the restriction $F_{|U_i} \to Y$ is u.a.n.q.c. for each .$i \in I$.

Proof. *Necessity***.** Suppose that *F* is u.a.n.q.c.. Let $i \in I$ and *x*∈ U_i and *V* be any regular open set of *Y* containing $(F_{|U_i})(x)$ and having *N*-closed complement. Since *F* is u.a.n.q.c. and $(F_{|U_i})(x) = F(x)$, by Lemma 2 there exists $U_o \in SO(X,x)$ such that $F(U_o) \subset V$. Let *U*=$U_o \cap U_i$. Then by Lemma 2 [20] we have *U*∈ $SO(U_i,x)$ and $(F_{|U_i})(U) \subset V$. It follows from Theorem 2 that $F_{|U_i}$ is u.a.n.q.c..

Sufficiency. Let *x*∈*X* and *V* be any regular open set containing *F(x)* and having *N*-closed complement. There exists $i \in I$ such that x∈ U_i and $(F_{|U_i})(x) = F(x) \subset V$. Since $F_{|U_i}$ is u.a.n.q.c., there exists *U*∈ $SO(U_i,x)$ such that $(F_{|U_i})(U) \subset V$. Since $U_i \in \alpha(X)$, it follows from Theorem 2 and Theorem 1 of [18] that U∈ $SO(X,x)$. Moreover, we have *F(U)* $\subset V$.

Theorem 12. Let (X,τ)be a topological space and $\{U_i : i \in I\}$ a cover of *X* by α −open sets of (X,τ). A multifunction $F : (X,\tau) \to (Y,\sigma)$ is l.a.n.q.c. if and only if the restriction $F_{|U_i} \to Y$ is l.a.n.q.c. for each .$i \in I$.

Proof. The proof is similar to proof of Theorem 5.

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Unele caracterizări ale multifuncțiilor superior/inferior aproape continue

Rezumat

În lucrarea [24], Rychlewicz a introdus noțiunea de multifuncție superior/inferior aproape cvasicontinuă ca o generalizare a noțiunii de multifuncție superior/inferior aproape continuă [23] şi a noțiunii de multifuncție superior/inferior aproape continuă [9]. Scopul lucrării noastre este de a obține noi teoreme de caracterizare a multifuncțiilor superior/inferior aproape continue.