

On a Representation of Mean Residual Life

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Abstract

In this paper, we obtain simple expressions for the mean residual life in terms of the failure rate of certain classes of distributions which subsume many of the standard cases. Several results in the literature can be obtained using our approach.

Key words: Mean Residual Life, distribution, failure rate

Introduction

In life testing situations, the expected additional lifetime given that a component has survived until time t is a function of t , called the mean residual life. More specifically, if the random variable X represents the life of a component, then the mean residual life is given by $m(t) = E(X - t | X > t)$.

Background and Definitions

Let $F: [0, \infty) \rightarrow [0, \infty)$ be a non decreasing, right continuous function with

$$F(0) = 0, \lim_{x \rightarrow \infty} F(x) = 1,$$

and let ν denote the induced Lebesgue – Stieljes measure. (Equivalently, let ν be a probability measure on $[0, \infty)$ and let F be a cumulative distribution function of ν). If X is a nonnegative random variable representing the life of a component having distribution function F , the mean residual life is defined by

$$m(t) = E(X - t | X > t) = \frac{1}{\bar{F}(t)} \int_t^\infty (x - t) d\nu(x), \quad t \geq 0,$$

where $\bar{F} = 1 - F$ is the so-called survival function. Writing $x - t = \int_t^x du$ and employing Tonelli's theorem yields the equivalent formula

$$m(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \int_t^x du d\nu(x) = \frac{1}{\bar{F}(t)} \int_t^\infty \int_u^\infty d\nu(x) du = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(u) du, \quad (1)$$

which is sometimes also used as a definition. The cumulative hazard function may be defined by $R = -\log \bar{F}$.

Then (1) implies that:

$$m(t) = \int_0^{\infty} \exp\{R(t) - R(t+x)\} dx. \quad (2)$$

If F (equivalently, v) is also absolutely continuous, then the probability density function f and the failure rate (hazard function) r are defined almost everywhere by $f = F'$ and $r = \frac{f}{F} = R'$ respectively, and then

$$R(t) = -\log \bar{F}(t) = -\int_0^t \frac{d\bar{F}(x)}{\bar{F}(x)} = \int_0^t r(x) dx. \quad (3)$$

In view of (2) and (3), we have expressed m in terms of r , albeit somewhat indirectly.

Ideally, we'd like to express the mean residual life in terms of known function of the failure rate and its derivatives without the use of integrals. In any case, it is useful to have alternative representation of the mean residual life. We note that the converse problem, that of expressing the failure rate in terms of the mean residual life and its derivatives is trivial, for (1) and (3) imply that

$$m'(t) = r(t)m(t) - 1. \quad (4)$$

Ultimately Increasing Failure Rate Distributions

Consider the class of distribution whose failure rate is ultimately increasing. More specifically, the failure rate should be strictly increasing from some point onward. Obviously, the important class of lifetime distributions having a bathtub-shaped failure rate with, change points $0 \leq t_1 \leq t_2 < \infty$ (i.e. for which the failure rate is strictly decreasing on the interval $[0, t_1]$, constant on $[t_1, t_2]$ and strictly increasing on $[t_2, \infty)$) constitutes a proper subclass of the distributions we consider here. We'll see that if the failure rate is strictly increasing from some point onward, then under certain additional conditions the mean residual life can be expanded in terms of Gaussian probability functions.

Notation. Our conventions regarding the Bachmann-Landau O -notation, the Vinogradov

\ll - notation, the symbol $O(h(t))$, $t \rightarrow \infty$, denotes an unspecified function g for which there exist positive real numbers t_0 and B such that $|g(t)| \leq B|h(t)|$ for all real $t > t_0$. For such g we write $g(t) \ll h(t)$ or $g(t) = O(h(t))$. The notation $g(t) = o(h(t))$, $t \rightarrow \infty$, means that for every real $\varepsilon > 0$, no matter how small, there exists a positive real number t_0 such that $|g(t)| \leq \varepsilon|h(t)|$ whenever $t > t_0$.

Theorem 1. Suppose that from some point onward, the failure rate r increases (strictly) without bound. Suppose further that for some positive integer n , the $n-1$ derivative is continuous and satisfies

$$|r^{(n-1)}(t+x)| \ll |r^{(n-1)}(t)| \quad t \rightarrow \infty \quad (5)$$

uniformly in x for $0 \leq x \leq \min(1, |r''(t)|^{-1/3})$ and $r^{(j)}(t) \ll \max(1, |r''(t)|^{j/3})$, $t \rightarrow \infty$, $3 \leq j \leq n-1$.

Finally, suppose there exists a positive real number ε such that for each integer j in the range $3 \leq j \leq n$,

$$r^{(j-1)}(t) = o((r(t))^{j-j\varepsilon}), \quad t \rightarrow \infty. \quad (6)$$

Then we have the following expression for the mean residual life:

$$m(t) = \sum_{k=0}^{n-1} b_k(t) \varphi_k(t) + o((r(t))^{-1-n\varepsilon}), t \rightarrow \infty. \quad (7)$$

where the coefficients $b_k(t)$ are given by the formal power series identity:

$$\sum_{k=0}^{\infty} b_k x^k = \exp\left\{-\sum_{k=3}^{\infty} r^{(k-1)}(t) \frac{x^k}{k!}\right\} \quad (8)$$

and

$$\begin{aligned} \varphi_k(t) &= \int_0^{\infty} x^k \exp\left\{-xr(t) - \frac{1}{2}x^2r'(t)\right\} dx \\ &= (-1)^k \sqrt{\frac{2\pi}{r'(t)}} \left[\frac{\partial^k}{\partial p^k} \left(1 - \Phi\left(\frac{p}{\sqrt{r'(t)}}\right) \right) \exp\left\{\frac{p^2}{2r'(t)}\right\} \Big|_{p=r(t)} \right]. \end{aligned} \quad (9)$$

Here,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{v^2}{2}} dv \quad (10)$$

is the Gaussian probability function, i.e. the cumulative distribution function of the standard normal distribution.

Before proving Theorem 1, we make some preliminary remarks and give two illustrative examples. First, if $r''(t) = 0$, then the uniformity condition on x in (5) should be interpreted as $0 \leq x \leq 1$. Next, observe that the hypothesis (6) becomes more restrictive as ε increases. In particular $\varepsilon > 1$ implies $\lim_{t \rightarrow \infty} r^{(j-1)}(t) = 0$. Of course, as ε increases, the error term in (7) decreases. On the other hand, if $0 < \varepsilon < 1$, then $r^{(j-1)}(t)$ does not necessarily approach zero, but the correspondingly weaker hypothesis implies a weaker conclusion (larger error term). In any case, since r increases without bound, the error term tends to zero as $t \rightarrow \infty$. Additionally, if r is infinitely differentiable we may let $n \rightarrow \infty$ in (3.3) to obtain the *convergent* infinite series expansion

$$m(t) = \sum_{k=0}^{\infty} b_k(t) \varphi_k(t),$$

valid for all sufficiently large values of t . (More specifically, for those t for which $r(t) > 1$.)

In general, however, we do not assume the failure rate has infinitely many derivatives; n is fixed and the generating function (8) is a formal power series. Expanding (8) to compute $b_k(t)$ in terms of $r(t)$ and its derivatives shows that if r has only $n - 1$ derivatives, then $b_k(t)$ is undefined if $k \geq n$. Differentiating (8) leads to the recurrence

$$b_{k+1}(t) = -\frac{1}{k+1} \sum_{j=2}^k \frac{r^{(j)}(t)}{j!} b_{k-j}(t), k \geq 2, \quad (11)$$

from which the coefficients $b_k(t)$ may be successively determined, starting with the initial values $b_0(t) = 1, b_1(t) = b_2(t) = 0$. On the other hand, an application of the multinomial theorem yields the explicit representation

$$b_k(t) = \sum_{p=0}^{\lfloor \frac{k}{3} \rfloor} (-1)^p \sum \prod_{j \geq 2} \frac{1}{\alpha_j!} \left(\frac{r^{(j)}(t)}{(j+1)!} \right)^{\alpha_j}, \quad (12)$$

in which $[k/3]$ is the greatest integer not exceeding $k/3$ and inner sum is over all non-negative integers α_2, α_3 , such that $\sum_{j \geq 2} \alpha_j = p$ and $\sum_{j \geq 2} (j+1)\alpha_j = k$.

Finally, we note that the functions φ_k of (9) may also give more explicitly. By setting $a = r(t)$ and $b = 2r'(t)$ in Lemma 1 below, we find that

$$\begin{aligned} \varphi_k(t) &= (-1)^k \left(\frac{2}{r'(t)} \right)^{\frac{k+1}{2}}, \\ x \left\{ \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2h} \lambda^{\frac{k}{2}-h} \Gamma\left(h + \frac{1}{2}\right) \left((1 - \Phi(\sqrt{2\lambda}))e^\lambda + 1/2 \sum_{j=0}^{h-1} \frac{\lambda^{j+1/2}}{\Gamma(j+\frac{3}{2})} \right), \right. \\ &\quad \left. - 1/2 \sum_{h=0}^{\lfloor k/2 \rfloor} h! \binom{k}{2h+1} \lambda^{\frac{k-1}{2}-h} \sum_{j=0}^h \frac{\lambda^j}{j!} \right\}, \end{aligned}$$

where $\lambda = (r(t))^2 / 2r'(t)$, $\Gamma\left(h + \frac{1}{2}\right) = \pi^{\frac{1}{2}} \prod_{j=1}^h \left(j - \frac{1}{2}\right)$, and Φ denotes the Gaussian probability function (10).

Lemma 1. Let a be a real number, let b be a positive real number, and let k be a non-negative integer. Then

$$\begin{aligned} &\int_0^\infty x^k \exp\{-ax - bx^2\} dx = \\ &= (-1)^k b^{-\frac{k+1}{2}} \left\{ \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2h} \lambda^{\frac{k}{2}-h} \Gamma\left(h + \frac{1}{2}\right) \left((1 - \Phi(\sqrt{2\lambda}))e^\lambda + \frac{1}{2} \sum_{j=0}^{h-1} \frac{\lambda^{j+\frac{1}{2}}}{\Gamma\left(j + \frac{3}{2}\right)} \right) \right. \\ &\quad \left. - 1/2 \sum_{h=0}^{\lfloor k/2 \rfloor} h! \binom{k}{2h+1} \lambda^{\frac{k-1}{2}-h} \sum_{j=0}^h \frac{\lambda^j}{j!} \right\}, \end{aligned}$$

where $\lambda = a^2/4b$.

Proofs of Theorem 1 and Lemma 1 are relegated to the next section.

Proofs

Proof of Theorem 1. Since the failure rate is strictly increasing from some point onward, there exist $t_0 \geq 0$ such that $r'(t) > 0$ for all $t \geq t_0$. Also, since $\lim_{t \rightarrow \infty} r(t) = \infty$, there exist $t_1 \geq 0$ such that $r(t) \geq 1$ for $t \geq t_1$. Now let $t \geq \max(t_0, t_1)$, $\delta = \delta(t) = \min(1, 1/\sqrt[3]{|r''(t)|})$, and set

$$I(t) := \int_0^\delta \exp\{R(t) - R(t+x)\} dx, \quad J(t) := \int_\delta^\infty \exp\{R(t) - R(t+x)\} dx$$

so that $m(t) = I(t) + J(t)$. We have

$$\begin{aligned} J(t) &\geq \int_\delta^\infty r(t+x) \exp\{R(t) - R(t+x)\} dx \\ &= \exp\{R(t) - R(t+\delta)\} \\ &= \exp\left\{-\delta r(t) - \int_0^\delta x r'(t+\delta-x) dx\right\}. \\ &\leq \exp\{-\delta r(t)\}. \end{aligned}$$

But $r'' = o(r^{3-3\varepsilon})$. By definition of, it follows that from some point onward we must have $\delta r \geq \min(r, r^\varepsilon)$. Therefore, if we set $v = \min(1, \varepsilon)$ then $v > 0$ and

$$J(t) \leq \exp\{-(r(t))^v\}, \quad (13)$$

for all sufficiently large values of t .

Next, we write

$$\begin{aligned} I(t) &= \int_0^\delta \exp\left\{-\sum_{k=1}^{n-1} r^{(k-1)}(t) \frac{x^k}{k!} - \frac{1}{(n-1)!} \int_0^x u^{n-1} r^{(n-1)}(t+x-u) du\right\} dx \\ &= \int_0^\delta \left(\sum_{k=0}^{n-1} b_k(t) x^k + E_n(x, t)\right) \exp\left\{-xr(t) - \frac{1}{2}x^2 r'(t)\right\} dx, \end{aligned}$$

where

$$\begin{aligned} E_n(x, t) &= \exp\left\{-\sum_{k=3}^{n-1} r^{(k-1)}(t) \frac{x^k}{k!} - \frac{1}{(n-1)!} \int_0^x u^{n-1} r^{(n-1)}(t+x-u) du\right\} - \sum_{k=0}^{n-1} b_k(t) x^k \\ &= O\left(x^n \max \prod_{j=2}^{n-1} |r^{(j)}(t)|^{\alpha_j}\right), \end{aligned}$$

and the maximum is taken over all non negative integers α_j satisfying $\sum_{j=2}^{n-1} (j+1)\alpha_j = n$.

In view of the fact that $r^{(j-1)} = o(r^{j-j\varepsilon})$ for $3 \leq j \leq n$, it follows that

$$E_n(x, t) = o(x^n (r(t))^{n-n\varepsilon}), \quad 0 \leq x \leq \delta.$$

If we now write

$$\begin{aligned} I(t) &= \sum_{k=0}^{n-1} b_k(t) \int_0^\infty x^k \exp\left\{-xr(t) - \frac{1}{2}x^2 r'(t)\right\} dx \\ &\quad - \sum_{k=0}^{n-1} b_k(t) \int_\delta^\infty x^k \exp\left\{-xr(t) - \frac{1}{2}x^2 r'(t)\right\} dx \\ &\quad + \int_0^\delta E_n(x, t) \exp\left\{-xr(t) - \frac{1}{2}x^2 r'(t)\right\} dx, \end{aligned}$$

then we find that

$$I = \sum_{k=0}^{n-1} b_k \varphi_k + \sum_{k=0}^{n-1} O(b_k r^k e^{-\delta r}) + o\left(r^{n-n\varepsilon} \int_0^\delta x^n e^{-xr} dx\right).$$

The hypotheses on r and the definition of the coefficients b_k imply that $b_k = O(r^{k-k\varepsilon})$. from the derivation of the estimate (4.1) for J we recall that $\exp(-\delta r) \leq \exp(-r^v)$, at least from some point onward. Finally, as

$$\int_0^\delta x^n e^{-xr} dx \leq \int_0^\infty x^n e^{-xr} dx = n! r^{-n-1}.$$

It follows that

$$I = \sum_{k=0}^{n-1} b_k \varphi_k + o(r^{-1-n\varepsilon}). \quad (14)$$

Since $m = I + J$, combining (13) and (14) gives the stated result for m .

To complete the proof, it remains only to establish the asserted evaluation of the integrals φ_k . But this is readily obtained by completing the square in the exponential and differentiating under the integral.

Proof of Lemma. By a straightforward change of variables, we find that

$$\begin{aligned} 2b^{\frac{k+1}{2}} \int_0^\infty x^k \exp\{-ax - bx^2\} dx &= e^\lambda \int_\lambda^\infty (t^{\frac{1}{2}} - \lambda^{\frac{1}{2}})^k e^{-t} t^{-1/2} dt \\ &= e^\lambda \sum_{h=0}^k \binom{k}{h} (-1)^{k-h} \lambda^{\frac{k-h}{2}} \Gamma\left(\frac{h+1}{2}, \lambda\right), \end{aligned} \quad (15)$$

where

$$\Gamma(\alpha, \lambda) = \int_\lambda^\infty t^{\alpha-1} e^{-t} dt \quad (16)$$

is the incomplete gamma function. If we integrate (16) by parts and then divide both sides of the result by $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, we obtain the recurrence formula

$$\frac{\Gamma(\alpha + 1, \lambda)}{\Gamma(\alpha + 1)} = \frac{\lambda^\alpha e^{-\lambda}}{\Gamma(\alpha)} + \frac{\Gamma(\alpha, \lambda)}{\Gamma(\alpha)},$$

which can be iterated to give

$$\frac{\Gamma(\alpha+k, \lambda)}{\Gamma(\alpha+k)} = e^{-\lambda} \sum_{h=0}^{k-1} \frac{\lambda^{\alpha+h}}{\Gamma(\alpha+h+1)} + \frac{\Gamma(\alpha, \lambda)}{\Gamma(\alpha)}, \quad (17)$$

valid for any non-negative integer k . In particular, when $\alpha = 0$,

$$\frac{\Gamma(k, \lambda)}{\Gamma(k)} = e^{-\lambda} \sum_{h=0}^{k-1} \frac{\lambda^h}{h!}. \quad (18)$$

Equation (18) is valid for all positive integers k if $\lambda \geq 0$; it is also valid when $k = 0$ if $\lambda > 0$.

Substituting $\alpha = \frac{1}{2}$ in (17) yields

$$\frac{\Gamma(k+\frac{1}{2}, \lambda)}{\Gamma(k+\frac{1}{2})} = e^{-\lambda} \sum_{h=0}^{k-1} \frac{\lambda^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})} + \frac{\Gamma(\frac{1}{2}, \lambda)}{\Gamma(\frac{1}{2})} = e^{-\lambda} \sum_{h=0}^{k-1} \frac{\lambda^{k+\frac{1}{2}}}{\Gamma(k+\frac{3}{2})} + 2(1 - \Phi(\sqrt{2\lambda})). \quad (19)$$

Using (18) and (19), we get the stated result from (15).

Applications

We provide two examples indicating how Theorem 1 may be applied.

Example 1. Consider a linear failure rate of the form

$$r(t) = \alpha + \beta t, \quad \beta > 0. \quad (20)$$

The motivation and application of (20) to analyzing various data sets has been demonstrated by Kodlin (1967) and Carbone et al. (1967). Statistical inference related to the linear failure rate model, has been studied Bain (1947), Shaked (1947) and more recently by Sen and Bhattacharya (1995). For this model, the hypotheses of Theorem 1 are trivially satisfied for any *positive* integer n and *any* positive real number. Since r'' vanishes identically in this case, we see that $b_k(t) = 0$ for $k > 0$ in (17) and in fact we have the extract result

$$m(t) = \int_0^{\infty} \exp\left\{-\left(a + \beta t\right)x - \frac{\beta x^2}{2}\right\} dx = \exp\left\{\frac{(a + \beta t)^2}{2\beta}\right\} \left(1 - \Phi\left(\frac{a + \beta t}{\sqrt{\beta}}\right)\right) \sqrt{\frac{2\pi}{\beta}}.$$

Example 2. Chen (2000) proposes the two-parameter distribution with cumulative distribution function given by

$$F(t) = 1 - \exp\{(1 - \exp(t^\beta))\lambda\}, \quad t > 0,$$

where $\lambda > 0$ and $\beta > 0$ are parameters. The corresponding hazard function is the ultimately strictly increasing function of t given by

$$r(t) = \lambda\beta t^{\beta-1} \exp(t^\beta), \quad t > 0. \quad (21)$$

It is straightforward, albeit somewhat tedious, to verify that Chen's failure rate (21) satisfies the hypotheses of Theorem 1 with $n > 2$ and $0 < \varepsilon \leq \frac{2}{3}$, clearly $\varepsilon = \frac{2}{3}$ is optimal here. Thus, with derivatives of r in (8) and (10) now coming (21), we see that the asymptotic formula

$$m(t) - \sum_{k=0}^{n-1} b_k \varphi_k(t) + o((r(t))^{-1-2n/3}), \quad t \rightarrow \infty$$

holds for *all* integers $n > 2$. In particular, as the error term in (7) tends to zero in the limit as $\rightarrow \infty$, we obtain the *convergent* infinite series representation

$$m(t) = \sum_{k=0}^{\infty} b_k(t) \varphi_k(t),$$

valid for *all* sufficiently large values of t . There is no need to work out the coefficients $b_k(t)$ explicitly in this case. One can simply use the recurrence (11) to generate them.

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O reprezentare a duratei de viață reziduală

Rezumat

În această lucrare se obține o expresie simplă pentru durata de viață reziduală utilizând rata de defectare. Rezultatele obținute se pot folosi pentru o clasă importantă de distribuții care caracterizează multe cazuri standard.