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The Convergence Theorem for Discrete-time Martingales

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Abstract

This paper gives an insight to the mathematical theory of the discrete-time martingale, including basic definitions and results. Still, the paper is focused on prooving a variant of the Convergence Theorem based on Doob's Inequality for martingales. The second part of this paper presents a proof of The Law of Large Numbers for discrete-times martingales, as an application of the Convergence Theorem.

Key words: *martingale, submartingale, stopping time*

Introduction

Let $(f_n)_{n\geq 0}$ be a real valued familiy of random variables defined on the probability space $(\Omega, \mathscr{F}, P)$ and let $(\mathscr{F}_n)_{n=1,2,\ldots}$ be a filtration (an increasing family of borelian fields contained in $\mathscr{F}$). The stochastic process $(f_n)_{n\geq 0}$ is said to be adapted to the family $(\mathscr{F}_n)_{n=1,2,\ldots}$ if, for all n, f_n is $\mathscr{F}_n$ - measurable.

Definition 1. The stochastic process $(f_n)_{n\geq 0}$, adapted to the filtration $(\mathscr{F}_n)_{n=1,2,\ldots}$, is a martingale (respectively submartingale) with respect to the filtration, if the following conditions hold:

1. f_n is P-integrable for all n;
2. For all $n > m$

$$E[f_n / \mathscr{F}_m] = f_m \text{ almost surely (a.s.),} \tag{1}$$

respectively

2'. For all $n > m$

$$E[f_n / \mathscr{F}_m] \geq f_m \text{ a.s.} - \text{ submartingale case.} \tag{2}$$

Remark 1. Due to the conditional expectation properties the above conditions may be written:

2. For all $A \in \mathscr{F}_n$, and for all $n > m$

$$\int_A f_n dP = \int_A f_m dP \Leftrightarrow E[f_n] = E[f_m]. \tag{3}$$

respectively

2'. For all $A \in \mathscr{F}_n$, and for all $n > m$

$$\int_A f_n dP \geq \int_A f_m dP \Leftrightarrow E[f_n] \geq E[f_m] \text{- submartingale case.} \qquad (4)$$

Stopping Times and Stopping Theorem

Definition 2. A random variable T defined on $(\Omega, \mathscr{F}, P)$ and valued in $\{0,1,....\infty\}$ is called a stopping time (or Markov time) with respect to the filtration $(\mathscr{F}_n)_{n=1,2,...}$, or simply $\mathscr{F}_n$ - stopping time if:

$$\text{for all } n \geq 0,\ \{T \leq n\} \in \mathscr{F}_n. \qquad (5)$$

or equivalent,

$$\text{for all } n \geq 0,\ \{T = n\} \in \mathscr{F}_n. \qquad (6)$$

To any stopping time T we may associate a borelian field $\mathscr{F}_T$ defined as following:

$$\mathscr{F}_T = \left\{ A \in \mathscr{B}\left(\bigcup_{n\geq 0} \mathscr{F}_n \right) : A \cap \{T \leq n\} \in \mathscr{F}_n, \forall n \geq 0 \right\}, \qquad (7)$$

equivalent to:

$$\mathscr{F}_T = \left\{ A \in \mathscr{B}\left(\bigcup_{n\geq 0} \mathscr{F}_n \right) : A \cap \{T = n\} \in \mathscr{F}_n, \forall n \geq 0 \right\}, \qquad (8)$$

where $\mathscr{B}\left(\bigcup_{n\geq 0} \mathscr{F}_n \right)$ is the borel field generated by $\bigcup_{n\geq 0} \mathscr{F}_n$. We consider that $\mathscr{F}_T$ represents the events known until the moment T.

Let $(f_n)_{n\geq 0}$ be a $\mathscr{F}_n$ - adapted family of random variables and let $T : \Omega \rightarrow \{0,1,...\infty\}$ be a stopping time. We define the random variable

$$f_T : \Omega \rightarrow R,\ f_T(\omega) = \begin{cases} 0 & , \quad T(\omega) = \infty \\ f_{T(\omega)}(\omega) & , \quad T(\omega) < \infty \end{cases}. \qquad (9)$$

We may easily observe that f_T is $\mathscr{F}_T$ - measurable.

Proposition 1. If $(f_n)_{n\geq 0}$ is $\mathscr{F}_n$ - adapted, then for all $A \in \mathscr{B}_R$ the random variable

$$D_A(\omega) = \begin{cases} \min\{n : f_n(\omega) \in A\} & , \quad \{n : f_n(\omega) \in A\} \neq \phi \\ \infty & , \quad \text{otherwise} \end{cases} \qquad (10)$$

is a $\mathscr{F}_n$ - stopping time (it is called the moment of the first entrance in A).

The following result will be given without proof. The ones who are interested in the proof may consult [2].

Theorem 1. (The Stopping Theorem for bounded stopping times) Let $(f_n)_{n\geq 0}$ be a $\mathscr{F}_n$ - submartingale (respectively $\mathscr{F}_n$ - martingale) and let $T_1 \leq T_2$ be two stopping times almost surely finite. Let us suppose that $T_1 \leq T_2 \leq m \in (0,\infty)$, then:

$$E[f_{T2} / \mathscr{F}_{T1}] \geq f_{T1}, \tag{11}$$

respectively

$$E[f_{T2} / \mathscr{F}_{T1}] = f_{T1}. \tag{12}$$

In particular,

$$E[f_{T2} / \mathscr{F}_{T1}] \geq E[f_{T1}], \tag{13}$$

respectively

$$E[f_{T2} / \mathscr{F}_{T1}] = E[f_{T1}]. \tag{14}$$

Doob's Maximal Lemma

Theorem 2. (Doob's Maximal Lemma) Let $(f_n)_{n\geq 0}$ be a nonnegative $\mathscr{F}_n$ - submartingale. Then for all $\varepsilon > 0$ and $n = 1,2,...$:

$$P\left(\max_{0\leq k\leq n} f_k \geq \varepsilon\right) \leq \frac{1}{\varepsilon} \int_{(\max\limits_{0\leq k\leq n} f_k \geq \varepsilon)} f_n \; dP \leq \frac{E[f_n]}{\varepsilon}. \tag{15}$$

Proof. The second inequality in (15) is obvious if considering the integral properties. In order to prove the first inequality, we define the random variable $f_n^* = \max\limits_{0\leq k\leq n} f_k$ and the stopping time $T = \min(k \geq 0, f_k \geq \varepsilon)$. According to Proposition 1 we have $T = D_{[\varepsilon,\infty)}$. We use Theorem 1 for the bounded stopping times $T \wedge n \leq n$, and also the equality $\{f_n^* \geq \varepsilon\} = \{f_{T\wedge n} \geq \varepsilon\}$. Considering that on the set $\{f_n^* \geq \varepsilon\}$ the inequalities $T \leq n$ and $f_T \geq \varepsilon$ hold, we obtain:

$$\varepsilon P(f_n^* \geq \varepsilon) = \varepsilon P(f_{T\wedge n} \geq \varepsilon) = \int_{\{f_{T\wedge n} \geq \varepsilon\}} \varepsilon dP \leq \int_{\{f_{T\wedge n} \geq \varepsilon\}} f_{T\wedge n} dP.$$

Due to Theorem 1 (13) we have:

$$\int_{\{f_{T\wedge n} \geq \varepsilon\}} f_{T\wedge n} dP \leq \int_{\{f_{T\wedge n} \geq \varepsilon\}} f_n dP = \int_{\{f_n^* \geq \varepsilon\}} f_n dP.$$

Finally, the relation (15) is provided as:

$$P(\{f_n^* \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_{\{f_n^* \geq \varepsilon\}} f_n dP.$$

Proposition 2. If $(f_n)_{n\geq 0}$ is a $\mathscr{F}_n$ -martingale so that $E(|f_n|^p) < \infty$ for all n and for a $p \geq 1$, then

$$\text{for all } \varepsilon > 0,\ P\left(\max_{0\leq k\leq n} |f_k| \geq \varepsilon\right) \leq \frac{E[|f_n|^p]}{\varepsilon^p}. \tag{16}$$

Proof. We may prove that if $(f_n)_{n\geq 0}$ is a $\mathscr{F}_n$-martingale, $p \geq 1$, and $E\left(|f_n|^p\right) < \infty$ for all n, then $\left(|f_n|^p\right)_{n\geq 0}$ is a $\mathscr{F}_n$-martingale. Then, by (15) we have:

$$P\left(\max_{0\leq k\leq n} |f_k| \geq \varepsilon\right) \leq P\left(\max_{0\leq k\leq n} |f_k|^p \geq \varepsilon^p\right) \leq \frac{E\left[|f_n|^p\right]}{\varepsilon^p}.$$

The Convergence Theorem and The Law of Large Numbers

Theorem 3 (The Convergence Theorem). Let $(f_n)_{n\geq 0}$ be a $\mathscr{F}_n$-martingale so that $\sup_{n\geq 0} E\left(f_n^2\right) < \infty$. Then there exists a random variable f such that $f_n \xrightarrow{a.s.} f$ and $f_n \xrightarrow{L^2} f$ and, moreover:

$$\text{for all} \quad n \geq 0,\ f_n = E\left[f / \mathscr{F}_n\right]. \tag{17}$$

Proof. We first notice that it is enough to prove the convergence in L^2 in order to obtain the a.s. convergence. Therefore, let us suppose that $f_n \xrightarrow{L^2} f$. Then, by applying Proposition 2 (16) for the martingale $(f_{n+k} - f_n)_{k\geq 1}$ and for $p = 2$, we have: for all $\varepsilon > 0$,

$$P\left(\max_{0\leq k\leq m} |f_{n+k} - f_n| \geq \varepsilon\right) \leq \frac{E\left[|f_{n+m} - f_n|^2\right]}{\varepsilon^2}.$$

As a consequence, we have:

$$P\left(\sup_k |f_{n+k} - f_n| \geq \varepsilon\right) = \lim_{m\to\infty} P\left(\max_{0\leq k\leq m} |f_{n+k} - f_n| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \lim_{m\to\infty} E\left[|f_{n+m} - f_n|^2\right].$$

Furthermore, $(f_n)_{n\geq 0}$ is Cauchy in L^2, so $\frac{1}{\varepsilon^2} \lim_{m\to\infty} E\left[|f_{n+m} - f_n|^2\right] \underset{n\to\infty}{\to} 0$.

We have obtained that $(f_n)_{n\geq 0}$ is Cauchy a.s., implying that $(f_n)_{n\geq 0}$ is convergent a.s.

We prove now the convergence in L^2. We may easily observe that for $(f_n)_{n\geq 0}$ a $\mathscr{F}_n$-martingale such that $E\left(|f_n|^2\right) < \infty$ for all n, then the following relations hold:

$$E\left[(f_{n+m} - f_n)^2 / \mathscr{F}_n\right] = E\left[\left(f_{n+m}^2 - f_n^2\right) / \mathscr{F}_n\right], \tag{18}$$

$$0 \leq \int (f_{n+m} - f_n)^2 \, dP = \int f_{n+m}^2 dP - \int f_n^2 \, dP. \tag{19}$$

Relation (19) implies that the sequence $\left(\int f_n^2\right)_{n\geq 0}$ is increasing. The sequence is also bounded (as $\sup_{n\geq 0} E\left(f_n^2\right) < \infty$), and thus convergent. From (19) we have $\lim_{m,n\to\infty} \int (f_{n+m} - f_n)^2 \, dP = 0$, meaning that $(f_n)_{n\geq 0}$ converges in L^2 to a limit f. This implies that $(f_n)_{n\geq 0}$ converges in L^1 to the limit f, meaning that $\int_A f_n dP \overset{n\to\infty}{\to} \int_A f dP$.

Particularly, because $(f_n)_{n\geq 0}$ is a $\mathscr{F}_n$-martingale, we have:

$$\text{For all } n \geq k,\ E[f_n / \mathscr{F}_k] = f_k, \tag{20}$$

$$\text{For all } n \geq k,\ E[f_n] = E[f_k]. \tag{21}$$

That means that:

$$\text{For all } A \in \mathscr{F}_k \text{ and } n \geq k,\ \int_A f_k dP = \int_A f_n dP \to \int_A f dP. \tag{22}$$

Hence:

$$\text{For all } A \in \mathscr{F}_k,\ \int_A f_k dP = \int_A f dP. \tag{23}$$

The last relation justifies relation (17): for all $n \geq 0$, $f_n = E[f / \mathscr{F}_n]$.

We remind the following lemma known as Kronecker's Lemma, that will be used in the further proof.

Lemma 1. Let $(a_n)_n$ be an increasing sequence of positive numbers having the limit ∞ and let $(x_n)_n$ be a sequence of real numbers such that $\sum_n \frac{x_n}{a_n}$ is convergent. Then:

$$\frac{1}{a_n}\sum_{k=1}^{n} x_k \underset{n\to\infty}{\to} 0. \tag{24}$$

Theorem 4. (The Law of Large Numbers for martingales) Let $(f_n)_{n\geq 0}$ be a $\mathscr{F}_n$-martingale such that:

$$\sum_{n=1}^{\infty} \frac{E\left(|f_n - f_{n-1}|^2\right)}{n^2} < \infty. \tag{25}$$

Then:

$$\frac{f_n}{n} \underset{n\to\infty}{\to} 0 \text{ a.s.}. \tag{26}$$

Proof. Without loss of generality, we may suppose that $f_0 = 0$. If defining the family of random variables $(h_n)_n$ as following: $h_o = 0$, for all $n \geq 1$, $h_n = \sum_{k=1}^{n} \frac{f_k - f_{k-1}}{k}$. We obtain $E[h_{n+1} - h_n / \mathscr{F}_n] = E\left[\frac{f_{n+1} - f_n}{n+1} / \mathscr{F}_n\right] = \frac{1}{n+1} E[(f_{n+1} - f_n) / \mathscr{F}_n] = 0$, implying that $(h_n)_{n\geq 0}$ is a martingale. Observe that $\frac{f_n}{n} = \frac{1}{n}\sum_{k=1}^{n} k(h_k - h_{k-1})$ and, due to Lemma 1, in order to show that $\frac{f_n}{n} \underset{n\to\infty}{\to} 0$ a.s. is enough to proove that $\sum_n \frac{n(h_n - h_{n-1})}{n} = \sum_n (h_n - h_{n-1})$ is a.s.

convergent, meaning that h_n is a.s. convergent. According to Theorem 3 it is enough to prove that $\sup_{n\geq 0} E\left(h_n^2\right) < \infty$. Indeed, as $E\left[h_n^2\right] = \sum_{k=1}^{n} \frac{E\left[\left|f_k - f_{k-1}\right|^2\right]}{k^2}$, then using (25), we obtain: $\sup_{n\geq 0} E\left(h_n^2\right) < \sum_{k=1}^{\infty} \frac{E\left[\left|f_k - f_{k-1}\right|^2\right]}{k^2} < \infty$ and the proof is complete.

Conclusion. For $(f_n)_{n\geq 0}$ a $\mathscr{F}_n$-discrete martingale, under the condition of $\sup_{n\geq 0} E\left(f_n^2\right) < \infty$ there exists a random variable f such that for all $n \geq 0$, $f_n = E\left[f / \mathscr{F}_n\right]$ and, moreover, $(f_n)_{n\geq 0}$ converges in L^2 and a.s. to f. The proof of the law of the large numbers presented before, is based on this result.

References

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Teorema de convergenţă pentru martingale cu timp discret

Rezumat

Articolul prezintă cateva rezultate importante ale teoriei martingalelor cu timp. In prima parte este demonstrată o variantă a Teoremei de Convergenţă a martingalelor cu timp discret pe baza inegalităţii lui Doob pentru martingale. A doua parte a acestui articol prezintă o demonstraţie a Legii Numerelor Mari, ca o consecinţă a Teoremei de Convergenţă.