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The Solution of the Characteristic Equation for Certain Damping Vibrations ($|\lambda_k| = p_k$)

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Abstract

For a dynamic system with n degrees of freedom, the present paper proposes the following conjecture: the roots of the characteristic equation for a dynamic system with damping which satisfies the stability conditions (they give the specific pulsations with damping) have the same modulus as the roots of the specific equation without any damping (the specific pulsations of the system with no damping). This statement is proved for a dynamic system with one degree of freedom.

Key words: *pulsations, damping, characteristic associate equation, proper pulsation without damping, proper pulsation with damping*

Introduction

In this introduction, we shall only recall some notations used in the theory of vibration. In the last part of the paper we shall formulate a conjecture on the modulus of the roots of the characteristic polynomial associate to a dynamic structure. We shall also prove it, at the end of this last part, for a dynamic system with one degree of freedom.

The dynamic calculation of structures is achieved in most cases on the basis of dynamic models with discrete masses which constitute systems with a finite number of freedom degrees. For such a system the equations of motion are written under the matrix form

$$\underset{\sim}{M} \cdot \ddot{\underset{\sim}{\eta}} + \underset{\sim}{B} \cdot \dot{\underset{\sim}{\eta}} + \underset{\sim}{R} \cdot \underset{\sim}{\eta} = \underset{\sim}{F}(t), \tag{1}$$

where $\underset{\sim}{M}$ is the inertial matrix ($\underset{\sim}{M} \in \mathcal{M}_{n,n}$ is a diagonal matrix and all the entries on the diagonal are positive real numbers); $\underset{\sim}{B}$ is the matrix of damping coefficients, ($\underset{\sim}{B} \in \mathcal{M}_{n,n}$); $\underset{\sim}{R}$ is the matrix of rigidity coefficients ($\underset{\sim}{R} \in \mathcal{M}_{n,n}$ is a positive definite matrix); $\underset{\sim}{\eta}$ is the motion vector ($\underset{\sim}{\eta} \in \mathcal{M}_{n,1}$); $\underset{\sim}{F}(t)$ is the vector of disturbance forces, where $\underset{\sim}{F}(t) = [F_1(t), F_2(t), \ldots, F_n(t)]^T$.

Structures Without Damping

Proper pulsations and proper oscillation vectors are determined from the matrix equation

$$\underset{\sim}{M} \cdot \ddot{\underset{\sim}{\eta}} + \underset{\sim}{R} \cdot \underset{\sim}{\eta} = 0, \tag{2}$$

which describes the free oscillations of the system without damping. Searching for solutions of the form $\underset{\sim}{\eta} = \underset{\sim}{A} \cos\ (pt)$ where $\underset{\sim}{A}$ is the vector of amplitudes in free stabilized vibration and p is the proper pulsation, out of the checking condition of equation (2) one obtains the matrix equation

$$\left(\underset{\sim}{R} - p^2 \underset{\sim}{M}\right) \cdot \underset{\sim}{A} = 0, \tag{3}$$

representing a homogeneous algebraic system.

The condition for (3) to have nontrivial solutions is: $\det\left(\underset{\sim}{R} - p^2 \cdot \underset{\sim}{M}\right) = 0$. (4)

Since $\underset{\sim}{R}$ is positive definite, the equation (4) always has n positive solutions. We shall also assume that its roots are simple:

$$p_1 < p_2 < \ldots < p_\alpha < \ldots < p_\beta < \ldots < p_n, \tag{5}$$

where p_1 represents the fundamental pulsation.

The characteristic equation which gives the proper pulsations of the system without damping (4) may be also written under the form

$$\Delta(\lambda) = \det\left(\underset{\sim}{R} + \lambda^2 \cdot \underset{\sim}{M}\right) = 0 \quad \text{with} \quad \lambda^2 = -p^2, \tag{6}$$

with $\Delta(\lambda)$ a polynomial of degree n in λ^2. When writing this equation in its explicit polynomial form, we shall use instead $\Delta(\lambda)$ the notation $\Delta^*(\lambda)$:

$$\Delta^*(\lambda) = a_0\left(\lambda^2\right)^n + \ldots + a_{n-1} \cdot \lambda^2 + a_n = 0. \tag{7}$$

The characteristic equation $\Delta(\lambda) = 0$ (6), and the characteristic associate equation $\Delta^*(\lambda) = 0$ (7), are identical: $\Delta(\lambda) = \Delta^*(\lambda),\ (\forall)\lambda \in C$. (8)

The determination of the coefficients $a_0, a_1, a_2, \ldots, a_n$ can be achieved by giving n+1 real distinct values to the parameter $\lambda = \lambda_k^*$ where $k \in \{0,1,2,\ldots,n\}$, $\lambda_i^* \neq \lambda_j^*$ for $i \neq j$.

We obtain the system

$$a_0\left(\lambda_k^{*2}\right)^n + a_1\left(\lambda_k^{*2}\right)^{n-1} + \ldots + a_{n-1}\lambda_k^{*2} + a_n = \Delta\left(\lambda_k^*\right), \text{ where } k = 0,1,2,\ldots,n, \tag{9}$$

which has always a unique solution. By solving it we obtain the coefficients of the characteristic equation (7), $a_0, a_1, \ldots, a_n$. Given $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the solutions of equation (7), then the equation (4) also has the solutions $p_1, p_2, \ldots, p_n$ with $\lambda_k = i \cdot p_k$ for $k = 1,2,\ldots,n$. (10)

representing the proper pulsations of the system without damping.

Determining the solutions of equation (7) is much more rapid than determining the solutions of equation (4) and can easily be modeled on a computer.

Structures with Certain Damping

In the case of structures with damping, the free vibrations are described by the matrix equation (1) in which $\underset{\sim}{F}(t)=0$. The solution to free vibrations is chosen under the form:

$$\underset{\sim}{\eta}1=\underset{\sim}{A}\cdot e^{\lambda\cdot t}, \tag{11}$$

where $\underset{\sim}{A}$ is the oscillation amplitudes vector in free vibrations.

Replacing in equation (1) for $\underset{\sim}{F}(t)=0$ we obtain a homogeneous system of equations, which, in order to have non-trivial solutions must fulfill the condition:

$$\delta(\lambda)=\det(\lambda^2\underset{\sim}{M}+\lambda\underset{\sim}{B}+\underset{\sim}{R})=0, \tag{12}$$

representing the characteristic equation which gives the proper pulsations of the system with damping. This is an algebraic equation of degree 2*n* in *λ*. When writing it in the polynomial form, equation (12) has the form

$$\delta(\lambda)=a_0\lambda^{2n}+a_1\lambda^{2n-1}+....+a_{2n-1}\cdot\lambda+a_{2n}=0. \tag{13}$$

and will be called a characteristic associate equation. The two equations (12) and (13) are identical:

$$\delta(\lambda)=\delta^*(\lambda) \qquad (\forall)\lambda\in C. \tag{14}$$

The roots of the equation (12) are complex roots and determining them is difficult, if not impossible, to achieve by means of the computer. To this aim we use equation (13).

The determination of the coefficients of the equation (13) is achieved by giving 2*n*+1 real to parameter $\lambda=\lambda_k^*$, $k\in\{0,1,2,...,2n\}$, where $\lambda_i^*\neq\lambda_j^*$ for $i\neq j$, and $i,j\in\{0,1,2,...,2n\}$.

We obtain the linear system:

$$a_0\left(\lambda_k^*\right)^{2n}+a_1\left(\lambda_k^*\right)^{2n-1}+...+a_{2n-1}\cdot\lambda_k^*+a_{2n}=\delta\left(\lambda_k^*\right) \tag{15}$$

for $k\in\{0,1,2,...,2n\}$. By solving the system, which has a unique solution, we obtain the $a_0,a_1,a_2,...,a_{2n}$ coefficients. In this way we obtain the characteristic associate equation for structures with damping. The solutions of equation (13) are complex conjugate numbers with negative real part, if the system is quasi-stable (see Definition 1 from below):

$$\lambda_{1,n+1}=\beta_1\pm i\cdot\gamma_1,\ \lambda_{2,n+2}=\beta_2\pm i\cdot\gamma_2,...,\lambda_{n,2n}=\beta_n\pm i\cdot\gamma_n,$$

where $\beta_1<0,\ \beta_2<0,...,\beta_n<0.$ (16)

The imaginary part of these roots represent the proper pulsations with damping, which are usually denoted with p_i^*. Consequently, $p_1^*=\gamma_1;\ p_2^*=\gamma_2;\ ...;p_n^*=\gamma_n$. (17)

Stability Condition for a Dynamic System

Definition 1. The dynamic structure characterized by the matrices $\underset{\sim}{M}$, $\underset{\sim}{B}$ and $\underset{\sim}{R}$ is called *quasi-stable* if the roots of equation (13), $\delta^*(\lambda)=0$ (are complex conjugate roots and) have the real part negative, meaning that they are all situated on the left side of the axis *Oy*.

In the theory of automatic systems, the notion of quasi-stability is equivalent to the notion of asymptotic stability.

If the equation $\delta^*(\lambda)=0$ has all the roots with the real part negative, then the polynomial $\delta^*(\lambda)$ is called Hurwitz polynomial.

The notion of quasi-stability is therefore equivalent with the condition that the characteristic polynomial $\delta(\lambda)$ is Hurwitzian.

To the characteristic polynomial $\delta^*(\lambda)$ it is attached the Hurwitz matrix

$$H_{2n}=\begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots 0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{2n} & a_{2n-1} & a_{2n-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 & a_{2n} \end{bmatrix} \tag{18}$$

and the submatrices H_k for $k=1,2,...,2n-1,$ (19)

where H_k contains the first *k* lines and *k* columns from the matrix H_{2n}.

The following result is well known in the mathematical literature.

Theorem 1. The characteristic polynomial $\delta(\lambda)$ is Hurwitzian if and only if

$$a_0>0,\ a_1>0,...,a_{2n}>0;\quad \det\ H_1>0;\quad \det H_2>0;\ ....\det\ H_{2n}>0\,. \tag{20}$$

Conditions (20) are necessary and sufficient so as the characteristic polynomial $\delta^*(\lambda)$ have the roots situated within the semiplane $\mathrm{Re}(\lambda)<0$, but it is not specified that those roots can not be real and negative. That, for the structure, would mean a non-periodical movement.

We are interested in finding some sufficient conditions on the coefficients of the Hurwitzian polynomial $\delta^*(\lambda)$, so that it should not have real negative roots.

In [6], the following conditions were introduced:

$$\begin{aligned} & a_0>0, a_2>0, ...a_{2n-2}>0, a_{2n}>0 \quad \text{and} \\ & a_1^2<2\cdot a_0\cdot a_2 \\ & a_3^2<a_2\cdot a_4 \\ & a_5^2<a_4\cdot a_6 \\ & . \\ & . \\ & a_{2n-3}^2<a_{2n-4}\cdot a_{2n-2} \\ & a_{2n-1}^2<2a_{2n-2}\cdot a_{2n} \end{aligned} \tag{21}$$

It was proved in [6] that if these conditions are fulfilled, then the equation $\delta(\lambda)=0$ has no real roots and, implicitly, no real negative roots.

We shall introduce now the notion of stable dynamic system (stable dynamic structure).

Definition 2. The dynamic system characterized by matrices $\underset{\sim}{M}$, $\underset{\sim}{B}$ and $\underset{\sim}{R}$ is *stable* if it is quasi-stable and the characteristic equation (13) has no real negative roots.

This definition becomes equivalent to the next one taking into account the notion of Hurwitzian polynomial.

The dynamic system is stable if and only if the characteristic polynomial $\delta^*(\lambda)$ is Hurwitz and if the equation $\delta^*(\lambda)=0$ has no real negative roots.

So the following statement is true.

Theorem 2. If the characteristic equation $\delta(\lambda)=0$ accomplishes the conditions (20) and (21) then the dynamic system is stable.

It is important to give, for systems with damping vibrations, sufficient conditions in order that $|\lambda_k| = p_k$ for $k \in \{1,2,...,n\}$, where λ_k is a solution to the equation (13) and p_k a solution to equation (4) representing the proper pulsation of the system without damping. There results

$$p_k = \sqrt{\beta_k^2 + \gamma_k^2} = \sqrt{\beta_k^2 + \left(p_k^*\right)^2} \quad \text{for} \quad \mathrm{k} \in \{1,2,...,\mathrm{n}\}. \tag{22}$$

We propose the following conjecture: $|\lambda_k| = p_k$ for $k \in \{1,2,...,n\}$ if the polynomial $\delta^*(\lambda)$ verifies the conditions (20) and (21), which means that the dynamic system is stable to vibrations.

The statement is true for systems with one-degree-of-freedom. In this case the equation (4) has the form:

$$-m \cdot p^2 + d = 0, \tag{23}$$

where *m* is the mass of the system and *d* is the rigidity coefficient. The roots of this equation are:

$$p_{1,2} = \pm\sqrt{\frac{d}{m}} \tag{24}$$

and the specific pulsation of the system without damping is $p_1 = \sqrt{\frac{d}{m}}$. (25)

If the system has the damping *b*, then equation (13) becomes:

$$\delta(\lambda) = \delta^*(\lambda) m \cdot \lambda^2 + b \cdot \lambda + d = 0. \tag{26}$$

The stability conditions (20) and (21) are: $b^2 < 2 \cdot m \cdot d$ and $m > 0$, $b > 0$, $d > 0$. (27)

The roots of the equation (26) are:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot m \cdot d}}{2 \cdot m}. \tag{28}$$

As $b^2 - 4 \cdot m \cdot d < b^2 - 2 \cdot m \cdot d < 0$, it follows that:

$$\lambda_{1,2} = \frac{-b}{2 \cdot m} + i\frac{\sqrt{4 \cdot m \cdot d - b^2}}{2 \cdot m} = \beta_1 + i \cdot \gamma_1, \tag{28'}$$

with $\beta_1 = -\frac{b}{2 \cdot m}$ and $\gamma_1 = \frac{\sqrt{4 \cdot m \cdot d - b^2}}{2 \cdot m}$.

Obviously, $|\lambda_1| = \sqrt{(\lambda_1)^2 + (\gamma_1)^2} = \sqrt{\frac{d}{m}} = p_1$, which means that this statement is true for systems with one-degree-of-freedom.

Conclusions

In this paper we have analyzed:

- the specific equation of the systems without damping noted with $\Delta^*(\lambda) = 0$ (7), with the roots $p_1.i$, $p_2.i$, …, $p_n.i$, with $p_1 < p_2 < \ldots < p_n$ (the specific pulsations of the system with no damping);
- the specific equation of the systems with damping, noted with $\delta^*(\lambda) = 0$ (13), which has the roots $\lambda_k = \beta_k \pm i \cdot \gamma_k$ with $k \in \{1,2,....,n\}$ (the imaginary part γ_k is represented by the specific pulsations with damping for $k \in \{1,2,....,n\}$).

If the coefficients $\delta^*(\lambda) = 0$ satisfy the stability conditions (20) and (21), then it is necessary to prove that, in general $|\lambda_k| = p_k$ for $k \in \{1,2,....,n\}$. This statement was demonstrated for the dynamic systems with one degree of freedom, but was not demonstrated on a general level.

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Rădăcinile ecuației caracteristice a unui sistem dinamic cu amortizare ($|\lambda_k| = p_k$)

Rezumat

Această lucrare propune următoarea conjectură pentru un sistem dinamic cu n grade de libertate: rădăcinile ecuației caracteristice pentru un sistem dinamic cu amortizare (care dau pulsațiile cu amortizare) care satisface condițiile de stabilitate au același modul ca rădăcinile ecuației caracteristice pentru sistemul dinamic fără amortizare (care dau pulsațiile sistemului fără amortizare). Această afirmație este demonstrată pentru sisteme cu un grad de libertate.