BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LXII No. 1/2010	109 - 112	Seria Matematică - Informatică - Fizică

A Refinement of Chen-Qi Inequality on the Harmonic Sum

Cristinel Mortici

Valahia University of Târgovişte, Department of Mathematics,
Bd. Unirii 18, 130082 Târgovişte, Romania
email: cmortici@valahia.ro

Abstract

The aim of this paper is to refine a double inequality of Chen and Qi stated in [The best bounds of harmonic sequence arXiv:math/0306233].

Keywords: *harmonic numbers, inequalities, gamma function, digamma function, approximations*

Introduction

The harmonic numbers

$$h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1)$$

are of great interest in analysis, since they are related to the sequence $h_n - \ln n$ convergent to the Euler-Mascheroni constant $\gamma = 0.577215...$. Furthermore, the harmonic sum has important connections to the Euler gamma function and other special functions. As an example, we mention the relation $\psi(n) = -\gamma + h_{n-1}$, where ψ is the digamma function [1, p.258].

In consequence, there is a huge literature about the harmonic numbers and related functions. Please refer to [2-12] and all the references therein.

Recently, Chen and Qi [Theorem 1][3] studied the variation of the function

$$\phi(x) = \frac{1}{\psi(x+1) - \ln x} - 2x \quad (x > 0)$$

to deduce as main result the following inequality

$$\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq h_n - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}} \quad (n \geq 1). \tag{1}$$

The involved constants $\frac{1}{1-\gamma}-2=0.36527...$ and $\frac{1}{3}=0.33333...$ are sharp in this inequality.

The proof provided in [3] is quite difficult. In order to prove the monotonicity of the function ϕ, Chen and Qi used estimates for the gamma and digamma function arising from some asymptotic expansions.

We give in this paper a simple and elementary proof of our new result which is an improvement of (1). We are convinced that our new approach is suitable for establishing many other similar results.

The Result

Inequalities involving h_n are useful to estimate the harmonic numbers for large values of n. Moreover, numerical computations show that the expression $h_n-\ln n-\gamma$ becomes close to the right-hand side of (1), when n approaches infinity. Moreover, (1) follows from the decreasing monotonicity of ϕ, that is $\phi(\infty)<\phi(n)\leq\phi(1)$, for every $n\geq 1$.

Motivated by these remarks, we prove the following improvement of Chen-Qi inequality (1).
Theorem 1. *For every integer* $n\geq 1$, *we have*

$$\frac{1}{2n+\frac{1}{3}+\frac{1}{18n}}<h_n-\ln n-\gamma<\frac{1}{2n+\frac{1}{3}+\frac{1}{32n}}.$$

Proof. The above inequalities are true for $n=1$, so we concentrate to prove the general case $n\geq 2$. In this sense, note that the sequences

$$a_n=h_n-\ln n-\gamma-\frac{1}{2n+\frac{1}{3}+\frac{1}{18n}},\quad b_n=h_n-\ln n-\gamma-\frac{1}{2n+\frac{1}{3}+\frac{1}{32n}}$$

are convergent to zero, so it suffices to show that a_n is strictly decreasing and b_n is strictly increasing. By denoting $a_{n+1}-a_n=f(n)$ and $b_{n+1}-b_n=g(n)$, we have to prove that $f<0$ and $g>0$, where

$$f(x)=\frac{1}{x+1}-\ln\left(1+\frac{1}{x}\right)-\frac{1}{2(x+1)+\frac{1}{3}+\frac{1}{18(x+1)}}+\frac{1}{2x+\frac{1}{3}+\frac{1}{18x}}$$

and

$$g(x)=\frac{1}{x+1}-\ln\left(1+\frac{1}{x}\right)-\frac{1}{2(x+1)+\frac{1}{3}+\frac{1}{32(x+1)}}+\frac{1}{2x+\frac{1}{3}+\frac{1}{32x}}.$$

We have

$$f'(x)=\frac{62\,808x+486\,792x^2+983\,448x^3+760\,752x^4+202\,176x^5+1849}{x(x+1)^2(6x+36x^2+1)^2(78x+36x^2+43)^2}$$

and

$$g'(x)=-\frac{P(x)}{x(x+1)^2(32x+192x^2+3)^2(416x+192x^2+227)^2},$$

where

$$P(x) = 99\,090\,432x^6 + 194\,248\,704x^5 - 91\,131\,904x^4 - 398\,864\,384x^3$$

$$-238\,946\,560x^2 - 26\,596\,992x - 463\,761.$$

All coefficients of the polynomial $P(x+2)$ are positive, so $f' > 0$ and $g' < 0$.

Finally, f is strictly increasing, g is strictly decreasing, with $f(\infty) = g(\infty) = 0$, so $f < 0$ and $g > 0$. The theorem is proved. ∎

References

1. Abramowitz, M, and Stegun, I. - *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover Publications, 1972

2. Artin, E. - *The Gamma Function*, New York, Holt,

3. Chen, C.-P., Qi, F. - The best bounds of harmonic sequence, arXiv:math/0306233, available online at: http://arxiv.org/abs/math/0306233

4. Mortici, C. - An ultimate extremely accurate formula for approximation of the factorial function, *Arch. Math. (Basel)*, 93, pp. 37-45, 2009

5. Mortici, C. - Product approximations via asymptotic integration, *Amer. Math. Monthly*, 117, pp. 434-441, 2010

6. Mortici, C. - New approximations of the gamma function in terms of the digamma function, *Appl. Math. Lett.*, 23, pp. 97-100, 2010

7. Mortici, C. - New sharp bounds for gamma and digamma functions, *Analele Stiintifice ale Universitatii Alexandru Ioan Cuza Iasi, Seria Noua Matematica*, 56, 2010, in press

8. Mortici, C. - Complete monotonic functions associated with gamma function and applications, *Carpathian J. Math.*, 25, pp. 186-191, 2009

9. Mortici, C. - Optimizing the rate of convergence in some new classes of sequences convergent to Euler's constant, *Anal. Appl. (Singap.)*, 8, pp. 99-107, 2010

10. Mortici, C. - Improved convergence towards generalized Euler-Mascheroni constant, *Appl. Math. Comput.*, 215, pp. 3443-3448, 2010

11. Mortici, C. - A class of integral approximations for the factorial function, *Comput. Math. Appl.*, 59, pp. 2053-2058, 2010

12. Mortici, C. - Best estimates of the generalized Stirling formula, *Appl. Math. Comput.*, 215, pp. 4044-4048, 2010

O rafinare a inegalităţii Chen-Qi despre suma armonică

Rezumat

Scopul acestui articol este de a rafina o dublă inegalitate datorată lui Chen şi Qi stabilită în [The best bounds of harmonic sequence arXiv:math/0306233].