BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LXII No. 1/2010	46 - 60	Seria Matematică - Informatică - Fizică

Properties of Weakly (τ, β) -Continuous Functions

Uğur Şengül

Marmara University, Faculty of Science and Letters, Department of Mathematics, 34722 Göztepe-İstanbul, Turkey
email: usengul@marmara.edu.tr

Abstract

Popa and Noiri have introduced the notion of weakly (τ, β)-continuous functions. In this paper we obtain several properties and new characterizations of weakly (τ, β)-continuous functions and show that for the many of known results more strong statements are true. So we improve and strengthen some of these results related to the weakly (τ, β)-continuous function.

Keywords: *weakly (τ, β)-continuous functions, β-open sets, β-regular sets, clopen sets, quasi-open sets, ultra Hausdorff spaces, co-βR-closed graphs*

Introduction and Preliminaries

Semi-open sets, preopen sets, α-sets, β-open sets play an important role for generalization of continuity in topological spaces. By using these sets several authors introduced and studied various modifications of continuity such as weak continuity, almost s-continuity [22], $p(\theta)$-continuity [6]. The notion of weakly (τ, m)-continuous functions were introduced and studied by Popa and Noiri [24] for unifying these three functions using minimal conditions. They also defined weakly (τ, β)-continuous functions as a special case. Recently Basu and Ghosh [5] have also used independently this function under the name of (θ, β)-continuous functions when studying β-closed spaces and they listed several properties of this functions without proof. We give new characterizations, improve and strengthen some of these results related to the weakly (τ, β)-continuous functions.

Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset S of (X, τ), $cl(S)$ and $int(S)$ represent the closure of S and the interior of S, respectively. A subset S of a space (X, τ) is said to be regular open [34] (resp. regular closed [34]) if $S = int(cl(S))$ (resp. $S = cl(int(S))$). A point x of X is called a θ-cluster [36] point of A if $cl(U) \cap A \neq \varnothing$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure [36] of A and is denoted by $cl_\theta(A)$. A set A is said to be θ-closed if $A = cl_\theta(A)$. The complement of a θ-closed set is said to be θ-open [19] .It is known that a subset U of a space X is θ-open if and only if for any $x \in U$, there exists an open set V in X such that $x \in V \subset cl(V) \subset U$. A subset S of a space (X, τ) is said to be semi-open [17] (resp. preopen [21], α-open [28], semi-preopen [4] or β-open [1]) if $S \subset cl(int(S))$ (resp. $S \subset int(cl(S))$, $S \subset int(cl(int(S)))$, $S \subset cl(int(cl(S)))$). The family of all semi-open (resp.preopen, α-open, β-open) subsets of X is denoted by $SO(X)$ (resp. $PO(X)$, $\alpha O(X)$, $\beta O(X)$) The complement of a semi-open (resp.preopen, α-open, β-open) set is said to be semi-closed (resp. preclosed, α-closed, β-closed). If S is a subset of a space X, then the β-closure of S, denoted by $\beta cl(S)$, is the smallest β-closed set containing S. The semiclosure (resp. preclosure, α-closure, β-closure) of S is similarly defined and is denoted by $scl(S)$ (resp. $pcl(S)$, $\alpha Cl(S)$, $\beta Cl(S)$). The β-interior of S, denoted by $\beta int(S)$, is the largest β-open set contained in S. A subset S is said to be β-regular if it is β-open and β-closed. The family of all β-closed (resp. β-regular) subsets of X is denoted by $\beta C(X)$ (resp. $\beta R(X)$) and the family of all β-open (resp. β-regular) subsets of X containing a point $x \in X$ is denoted by $\beta O(X, x)$ (resp. $\beta R(X, x)$). A point $x \in X$ is said to be in the β-θ-closure (=sp-θ-closure [27]) of A, denoted by $\beta cl_\theta(A)$, if $A \cap \beta cl(V) \neq \emptyset$ for every $V \in \beta O(X, x)$. If $\beta cl_\theta(A) = A$, then A is said to be β-θ-closed (=sp-θ-closed [27]). The complement of a β-θ-closed set is said to be β-θ-open (=sp-θ-open [27]).

The quasi-component [33] of a point $x \in X$ is the intersection of all clopen subsets of X which contain the point x. The quasi-topology τ_q on X is the topology having as base clopen subsets of (X, τ). The closure of each point in quasi-topology is precisely the quasi-component of that point. The open (resp. closed) subsets of the quasi-topology is called quasi-open [9] (resp. quasi-closed [9]). For a space (X, τ) the space (X, τ_q) is called by Staum [33] the ultraregular kernel of X and denoted by X_q for simplicity. A space (X, τ) is called ultraregular [33] if $\tau = \tau_q$.For a subset A of a space X, we define the quasi-interior (resp. quasi-closure) of A, denoted by $int_q(A)$ (resp. $cl_q(A)$), defined by $int_q(A) = \cup\{U$ is quasi-open:$U \subset A\}$, (resp. $cl_q(A) = \cap\{F$ is quasi-closed:$A \subset F\}$).

Lemma 1. [27] *Let A and B be any subsets of a space X. Then the following properties hold:*

(a) *For a subset A of a space X, $\beta cl_\theta(A) = \cap\{V : A \subset V$ and $V \in \beta R(X)\} =.\cap\{V : A \subset V$ and V is β-θ-closed$\}$*

(b) $x \in \beta cl_\theta(A)$ *if and only if* $A \cap V \neq \emptyset$ *for each* $V \in \beta R(X, x)$.

(c) *if* $A \subset B$ *then* $\beta cl_\theta(A) \subset \beta cl_\theta(B)$.

(d) $\beta cl_\theta(\beta cl_\theta(A)) = \beta cl_\theta(A)$.

(e)*intersection of an arbitrary family of β-θ-closed sets in X is β-θ-closed in X.*

(f) *A is β-θ-open if and only if for each $x \in A$, there exists $V \in \beta R(X, x)$ such that $x \in V \subset A$.*

(g) *If* $A \in \beta R(X)$ *then* A *is* β-θ*-closed and* β-θ*-open.*

(h)*If* $A \in \beta O(X)$ *then* $\beta cl(A) = \beta cl_\theta(A)$.

(i) $A \in \beta O(X)$ *if and only if* $\beta cl(A) \in \beta R(X)$.

(j)$A \in \beta C(X)$ *if and only if* $\beta int(A) \in \beta R(X)$.

Definition 1. [24] A function $f : X \to Y$ is weakly (τ, β)-continuous for each $x \in X$ and each $V \in \beta O(Y, f(x))$, there exists an open set U containing x such that $f(U) \subset \beta cl(V)$.

Theorem 2. [25] *For a function* $f : X \to Y$*, the following are equivalent:*

(a) f *is weakly* (τ, β)*-continuous.*

(b) *For each* $x \in X$ *and and each* $V \in \beta O(Y, f(x))$*, there exists an* α*-open set* U *containing* x *such that* $f(U) \subset \beta cl(V)$.

(c) $f^{-1}(V)$ *is* α*-open in* X *for every* β*-clopen set* V *of* Y.

(d) $f^{-1}(V)$ *is clopen in* X *for every* β*-clopen set* V *of* Y.

Theorem 3. [5] *For a function* $f : X \to Y$*, the following are equivalent:*

(a) f *is weakly* (τ, β)*-continuous.*

(b) *For each* $x \in X$ *and each filter base* F *on* X *converges to* x*, the filter base* $f(F)$ β-θ*-converges to* $f(x)$.

(c) *For each* $x \in X$ *and every net* (x_i) *in* X *converges to* x*,* $(f(x_i))$ β-θ*-converges to* $f(x)$.

Characterizations

Definition 2. [30] A subfamily m_X of the power set $\wp(X)$ of a nonempty set X is called a minimal structure (briefly m-structure) on X if $\varnothing \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty subset X with a minimal structure m_X on X. Each member of m_X is said to be m_X-open and the complement of m_X-open set is said to be m_X-closed.

Remark 1. *Let* (X, τ) *be a topological space. Then the families* τ*,* τ_q*,* $SO(X)$*,* $PO(X)$*,* $\alpha(X)$*,* $\beta(X)$ *(=*$\beta O(X)$*),* $SR(X)$*,* $\beta R(X)$ *are all* m*-structures on* X.

Definition 3. A function $f : (X, m_X) \to (Y, m_Y)$, where X and Y are nonempty sets with minimal structures m_X and m_Y, respectively,is said to be weakly M-continuous [25] (M-continuous [30]) at $x \in X$ if for each $V \in m_Y$ containing $f(x)$ there exists $U \in m_X$ containing x such that $f(U) \subset m_Y$-$Cl(V)$ (resp. $f(U) \subset V$). A function $f : (X, m_X) \to (Y, m_Y)$ is said to be weakly M-continuous (resp. M-continuous) if it has the property at each point $x \in X$.

Theorem 4. *For a function* $f : X \to Y$*, the following are equivalent:*

(a) f *is weakly* (τ, β)*-continuous.*

(b) *For each* $x \in X$ *and each* $V \in \beta R(Y, f(x))$*, there exists a clopen set* U *containing* x *such that* $f(U) \subset V$*.*

(c) *For each* $x \in X$ *and each* $V \in \beta R(Y, f(x))$*, there exists a quasi-open set* U *of* X *containing* x *such that* $f(U) \subset V$*.*

(d) $f : (X, \tau_q) \to (Y, \beta O(Y))$ *is weakly* M*-continuous.*

(e) $f^{-1}(V) \subset int_q(f^{-1}(\beta cl(V)))$ *for every* $V \in \beta O(Y)$*.*

(f) $cl_q(f^{-1}(\beta int(F))) \subset f^{-1}(F)$ *for every* $F \in \beta C(Y)$*.*

(g) $cl_q(f^{-1}(V)) \subset f^{-1}(\beta cl(V))$ *for every* $V \in \beta O(Y)$*.*

(h) $f(cl_q(A)) \subset \beta cl_\theta(f(A))$ *for each subset* A *of* X*.*

(i) $cl_q(f^{-1}(B)) \subset f^{-1}(\beta cl_\theta(B))$ for each subset B of Y.

Proof. (a)⇒(b): Let $x \in X$ and $V \in \beta R(Y, f(x))$. Then by Theorem 2, $f^{-1}(V)$ is clopen set containing x in X. Set $U = f^{-1}(V)$ this gives $f(U) \subset V$.

(b)⇒(c)⇒(a): These implications are clear from the definition of quasi topology.

(c)⇒(d)Let $x \in X$ and $V \in \beta R(Y, f(x))$.Then by (c) there exists a quasi-open set U containing x such that $f(U) \subset V$. Since every β-regular set is β-open, f is M-continuous, hence weakly M-continuous.

(d)⇒(a) Let $x \in X$ and $V \in \beta O(Y, f(x))$ then there exists a quasi-open set U containing x such that $f(U) \subset \beta cl(V)$. Since U is quasi open there exists an open set W in U containing x such that $f(W) \subset \beta cl(V)$ and by Definition 1 f is weakly (τ, β)-continuous.

(c)⇒(e): Let $V \in \beta O(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$ and $\beta cl(V) \in \beta R(Y, f(x))$ hence by (c), there exists a quasi-open set U of X containing x such that $f(U) \subset \beta cl(V)$. Then $x \in U \subset f^{-1}(\beta cl(V))$ and hence $x \in int_q(f^{-1}(\beta cl(V)))$.

(e)⇔(a): It follows from Theorem 3.2 of [25].

(e)⇒(f): Let $F \in \beta C(Y)$, then $Y - F \in \beta O(Y)$ and by (e), we have $f^{-1}(Y - F) \subset int_q(f^{-1}(\beta cl(Y - F)))$ i.e., $X - f^{-1}(F) \subset int_q(f^{-1}(\beta cl(Y - F))) = int_q(f^{-1}(Y - \beta int(F))) = X - cl_q(f^{-1}(\beta int(F)))$ Hence we obtain $cl_q(f^{-1}(\beta int(F))) \subset f^{-1}(F)$.

(f)⇔(a): It follows from Theorem 2.1 of [26].

(g)⇔(a) It follows from Theorem 3.4 of [25].

(a)⇒(h)⇒(i)⇒(a): It follows from Theorem 3.3 of [25]. □

Corollary 5. *For a function* $f : X \to Y$*, the following are equivalent:*

(a) f *is weakly* (τ, β)*-continuous.*

(b) *For each* $x \in X$ *and and each each* $V \in \beta R(Y, f(x))$*, there exists an open set* U *containing* x *such that* $f(cl(U)) \subset V$*.*

(c) $f^{-1}(V)$ *is* θ*-open and* θ*-closed in* X *for every* $V \in \beta R(Y)$*.*

(d) $f^{-1}(V) \subset int_\theta(f^{-1}(\beta cl(V)))$ *for every* β*-open* V *in* Y*.*

(e) $cl_\theta(f^{-1}(\beta int(V))) \subset f^{-1}(V)$ *for every* β*-closed* V *in* Y. $\in$

(f) $cl_\theta(f^{-1}(V)) \subset f^{-1}(\beta cl(V))$ *for every* β*-open* V *in* Y.

Corollary 6. *The following properties are equivalent for a function* $f : X \to Y$:

(a) f *is weakly* (τ, β)*-continuous.*

(b) *For each* $x \in X$ *and each* $V \in \beta O(Y, f(x))$*, there exists an open set* U *containing* x *such that* $f(int(cl(U))) \subset \beta cl(V)$.

(c) *For each* $x \in X$ *and each* $V \in \beta O(Y, f(x))$*, there exists a regular open set* U *containing* x *such that* $f(U) \subset \beta cl(V)$.

(d) $f^{-1}(V) \subset \delta\text{-}int(f^{-1}(\beta cl(V)))$ *for every* $V \in \beta O(Y)$.

(e) $\delta\text{-}cl(f^{-1}(\beta int(B))) \subset f^{-1}(B)$ *for every* $B \in \beta C(Y)$.

(f) $f(\delta\text{-}cl(A)) \subset \beta cl_\theta(f(A))$ *for every subset* A *of* X.

(g) $\delta\text{-}cl(f^{-1}(K)) \subset f^{-1}(\beta cl_\theta(K))$ *for every subset* K *of* Y.

(h) $\delta\text{-}cl(f^{-1}(V)) \subset f^{-1}(\beta cl(V))$ *for every subset* $V \in \beta O(Y)$.

Definition 4. A filter base $\mathcal{F}$ is said to be;

(a) β-θ-convergent [5] to a point x in X, if for any β-open set U containing x there exist $B \in \mathcal{F}$ such that $B \subset \beta cl(U)$,

(b) clopen convergent to a point x in X, if for any clopen set U containing x, there exist $B \in \mathcal{F}$ such that $B \subset U$.

Theorem 7. *A function* $f : X \to Y$ *is weakly* (τ, β)*-continuous if and only if for each point* $x \in X$ *and each filter base* $\mathcal{F}$ *in* X *that clopen converging to* x *the filter base* $f(\mathcal{F})$ *is* β*-*θ*-convergent to* $f(x)$.

Proof. Suppose that $x \in X$ and $\mathcal{F}$ is any filter base in X that clopen converges to x. By hypothesis for any β-open set V containing $f(x)$ there exists a clopen set U containing x in X such that $f(U) \subset \beta cl(V)$. Since $\mathcal{F}$ is clopen convergent to x in X then there exists $B \in \mathcal{F}$ such that $B \subset U$. It follows that $f(B) \subset \beta cl(V)$. This means that $f(\mathcal{F})$ is β-θ-convergent to $f(x)$.

Conversely, let x be a point in X and V be a β-open set containing $f(x)$. If we set $\mathcal{F} = \{U : U \text{ is clopen and } x \in U\}$, then $\mathcal{F}$ will be a filter base which clopen converges to x. So there exists $U \in \mathcal{F}$ such that $f(U) \subset \beta cl(V)$. This completes the proof. □

Definition 5. A net (x_i) in a space X, θ-converges [36] (resp. clopen converges [11], β-θ-converges [5]) to x if and only if for each open (resp. clopen, β-open,) set U containing x, there exists i_0 such that $x_i \in cl(U)$ (resp. $x_i \in U$, $x_i \in \beta cl(U)$) for all $i \geqq i_0$.

Lemma 8. *For a net* (x_i) *in a space* X*;*

(a) [8] *if* (x_i) *converges to* x*, then* (x_i) θ*-converges to* x .

(b) [11] *if* (x_i) *converges or* θ*-converges to* x*, then* (x_i) *clopen converges to* x*.*

Theorem 9. *For a function* $f : X \to Y$*, the following statements are equivalent:*

(a) f *is weakly* (τ, β)*-continuous.*

(b) *For each* $x \in X$ *and each net* (x_i) *in* X *which clopen converges to* x*, the net* $(f(x_i))$ β*-*θ*-converges to* $f(x)$*.*

(c) *For each* $x \in X$ *and each net* (x_i) *in* X *which* θ*-converges to* x*, the net* $(f(x_i))$ β*-*θ*-converges to* $f(x)$*.*

(d) *For each* $x \in X$ *and each net* (x_i) *in* X *which converges to* x*, the net* $(f(x_i))$ β*-*θ*-converges to* $f(x)$*.*

Proof.$(a) \Rightarrow (b)$ Let $x \in X$ and let (x_i) be a net in X such that (x_i) clopen converges to x. Let V be a β-open set containing $f(x)$. Since f is weakly (τ, β)-continuous and $\beta cl(V) \in \beta R(Y)$, there exists a clopen set U containing x such that $f(U) \subset \beta cl(V)$. Since (x_i) clopen converges to x, there exists i_0 such that $x_i \in U$ for all $i \geqq i_0$. Hence $f(x_i) \in \beta cl(V)$ for all $i \geqq i_0$.

$(b) \Rightarrow (c)$ Let $x \in X$ and let (x_i) be a net in X such that (x_i) θ-converges to x. By Lemma 8, (x_i) clopen converges to x. By (b), $(f(x_i))$ β-θ-converges to $f(x)$.

$(c) \Rightarrow (d)$ Let $x \in X$ and let (x_i) be a net in X such that (x_i) converges to x. By Lemma 8, (x_i) θ-converges to x. By (c), $(f(x_i))$ β-θ-converges to $f(x)$.

$(d) \Rightarrow (a)$ Suppose that f is not weakly (τ, β)-continuous. Then there exists $x \in X$ and a β-open set V containing $f(x)$ such that $f(U) \nsubseteq \beta cl(V)$ for all open U containing x. Consider the set $\{x_U : U$ is open set containing $x\}$. Then (x_U) converges to x but $(f(x_U))$ does not β-θ-converge to $f(x)$. $\square$

Proposition 10. [5] *A net* (x_i) *in a space* X*,* β*-*θ*-converges to* x *if and only if for each* β*-regular set* U *containing* x*, there exists* i_0 *such that* $x_i \in U$ *for all* $i \geqq i_0$*.*

By Theorem 9 and Proposition 10 we have the following theorem.

Theorem 11. *For a function* $f : X \to Y$*, the following are equivalent:*

(a) f *is weakly* (τ, β)*-continuous.*

(b) *If for each* $x \in X$ *and, a net* (x_i) *in* X *clopen converges to* x *then for each* $V \in \beta R(Y, f(x))$*, there exists* i_0 *such that* $f(x_i) \in V$ *for all* $i \geqq i_0$*.*

(c) *If for each* $x \in X$ *and, a net* (x_i) *in* X θ*-converges to* x *then for each* $V \in \beta R(Y, f(x))$*, there exists* i_0 *such that* $f(x_i) \in V$ *for all* $i \geqq i_0$*.*

(d) *If for each* $x \in X$ *and, a net* (x_i) *in* X *converges to* x *then for each* $V \in \beta R(Y, f(x))$*, there exists* i_0 *such that* $f(x_i) \in V$ *for all* $i \geqq i_0$*.*

Definition 6. *A function* $f : X \to Y$ *is called;*

(a) *weakly* α*-continuous* [23] *if for each* $x \in X$ *and each open set* V *of* Y *containing* $f(x)$*, there exists an* α*-open* U *of* X *containing* x *such that* $f(U) \subset cl(V)$*,*

(b) *θ-β-irresolute* [13] *(resp. weakly β*-irresolute [27]*) if for each $x \in X$ and each $V \in \beta O(Y, f(x))$, there exists $U \in \beta O(X, x)$ such that $f(\beta cl(U)) \subset \beta cl(V)$ (resp. $f(U) \subset \beta cl(V)$)*

(c) *slightly continuous* [14] *if $f^{-1}(V)$ is clopen in X for every clopen set V of Y.*

Theorem 12. *If functions $f : X \to Y$ and $g : Y \to Z$ satisfy each one of the following three properties, then the composition $g \circ f : X \to Z$ is weakly (τ, β)-continuous:*

(a) *f is weakly α-continuous and g is weakly (τ, β)-continuous.*

(b) *f is weakly (τ, β)-continuous and g is θ-β-irresolute.*

(c) *f is slightly continuous and g is weakly (τ, β)-continuous.*

Proof. (a) Let $x \in X$ and W be a β-open subset of Z containing $(g \circ f)(x)$. Since $\beta cl(W) \in \beta R(Z)$ and g is weakly (τ, β)-continuous, there exists a clopen set V of Y containing $f(x)$ such that $g(V) \subset \beta cl(W)$. Since f is weakly α-continuous there exists $U \in \alpha O(X, x)$ such that $f(U) \subset V$ and hence we obtain $(g \circ f)(U) \subset \beta cl(W)$. Therefore by Theorem 2, $g \circ f$ is weakly (τ, β)-continuous.

(b) Let $x \in X$ and W be a β-open subset of Z containing $(g \circ f)(x)$. Since g is θ-β-irresolute, there exists $V \in \beta O(Y, f(x))$ such that $g(\beta cl(V)) \subset \beta cl(W)$. Since f is weakly (τ, β)-continuous there exists an open sets U in X containing x such that $f(U) \subset \beta cl(V)$. This shows that $(g \circ f)(U) \subset \beta cl(W)$. Therefore $g \circ f$ is weakly (τ, β)-continuous.

(c) Let W be any β-regular subset of Z. Since g is weakly (τ, β)-continuous then by Theorem 2, the inverse image of W is clopen in Y. Since f is slightly continuous $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is clopen in X. Therefore $g \circ f$ is weakly (τ, β)-continuous. □

Corollary 13. *The composition of two weakly (τ, β)-continuous functions is weakly (τ, β)-continuous.*

Proof. By Remark 7.1 of [24] every weakly (τ, β)-continuous function is almost s-continuous and by [12] every almost s-continuous is almost continuous in the sense of Singal [32]. Since every almost continuous function is weakly α-continuous and slightly continuous [12, Corollary 5.1] result follows from Theorem 12. □

Separation Axioms and co-βR-closed Graphs

Definition 7. A space X is said to be;

(a) ultra T_0 [15] if for each pair of distinct points x and y of X, there exist a clopen set U containing one of the points x and y but not the other,

(b) ultra Hausdorff [33] if every two distinct points of X can be separated by disjoint clopen sets,

(c) ultranormal [33] if each pair of nonempty closed disjoint sets can be separated by disjoint clopen sets,

(d) clopen T_1 [10] if for each pair of distinct points x and y of X, there exist clopen sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$,

(e) β-T_2 [27] if for each pair of distinct points x and y in X, there exist β-open sets U and V of X containing x and y, respectively, such that $U \cap V = \varnothing$ (equivalently $\beta cl(U) \cap \beta cl(V) = \varnothing$),

(f) β-normal [35] if for any pair of disjoint closed sets A and B, there exist disjoint β-open sets U and V such that $A \subset U$ and $B \subset V$.

Remark 2. *Kohli and Singh proved that [15] ultra Hausdorff, clopen T_1, and ultra T_0 axioms are all equivalent.*

Definition 8. [25] A nonempty set X is with a minimal structure m_X, (X, m_X), is said to be m-Hausdorff if for each distinct points x, $y \in X$, there exist U, $V \in m_X$ containing x and y, respectively, such that $U \cap V = \varnothing$.

Theorem 14. *If $f : (X, \tau_q) \to (Y, m_Y)$ is a weakly M-continuous function and (Y, m_Y) is m-Hausdorff, then f has quasi-closed point inverses in X.*

Proof. Let $y \in Y$. We show that $f^{-1}(y) = \{x \in X : f(x) = y\}$ is quasi closed in X, or equivalently $A = \{x \in X : f(x) \neq y\}$ is quasi open in X. Let $x \in A$. Since $f(x) \neq y$ and (Y, m_Y) is m-Hausdorff, there exist disjoint m_Y-open sets V_1, V_2 such that $f(x) \in V_1$ and $y \in V_2$. Since $V_1 \cap V_2 = \varnothing$ by Lemma 3.2 of [24], we have m_Y-$Cl(V_1) \cap V_2 = \varnothing$. Thus $y \notin m_Y$-$Cl(V_1)$. Since f is weakly M-continuous function, there exists a quasi-open set U containing x such that $f(U) \subset m_Y$-$Cl(V_1)$. Now suppose that U is not contained in A. Then there exists a point $u \in U$ such that $f(u) = y$. Since $f(U) \subset m_Y$-$Cl(V_1)$, $y = f(u) \in m_Y$-$Cl(V_1)$. This is a contradiction. Therefore, $U \subset A$ and hence A is quasi-open in X. □

Corollary 15. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly (τ, β)-continuous and (Y, σ) is β-T_2 then f has quasi-closed point inverses in X.*

Theorem 16. I*f $f, g : X \to Y$ is weakly (τ, β)-continuous function and Y is β-T_2 , then $A = \{x \in X : f(x) = g(x)\}$ is quasi-closed in X.*

Proof. If $x \in X - A$, then it follows that $f(x) \neq g(x)$. Since Y is β-T_2, there exist $U \in \beta O(Y, f(x))$ and $V \in \beta O(Y, g(x))$ such that $\beta cl(U) \cap \beta cl(V) = \varnothing$. Since f and g are weakly (τ, β)-continuous there exists clopen sets G and H with $x \in G$ and $x \in H$ such that $f(G) \subset \beta cl(U)$ and $g(H) \subset \beta cl(W)$, set $O = G \cap H$. Then O is clopen, $f(O) \cap g(O) = \varnothing$ and $A \cap O = \varnothing$. Thus every point of $X - A$ has a clopen neighborhood disjoint from A. Hence $X - A$ is union of clopen sets or equivalently A is quasi-closed. □

Theorem 17. *If $f : X \to Y$ is weakly (τ, β)-continuous function and Y is β-T_2 , then $A = \{(x, y) \in X \times X : f(x) = f(y)\}$ is quasi-closed in $X \times X$.*

Proof. Let $(x, y) \in (X \times X) - A$, then it follows that $f(x) \neq f(y)$. Since Y is β-T_2, there exist $U \in \beta R(Y, f(x))$ and $V \in \beta R(Y, f(y))$ such that $U \cap V = \varnothing$. Since f is weakly (τ, β)-continuous $f^{-1}(U)$ and $f^{-1}(V)$ are clopen in X hence so is $f^{-1}(U) \times f^{-1}(V)$. Hence $(f^{-1}(U) \times f^{-1}(V)) \cap A = \varnothing$. Thus every point of $(X \times X) - A$ has a clopen neighborhood disjoint from A. Hence $(X \times X) - A$ is union of clopen sets or equivalently A is quasi-closed in $X \times X$. □

Definition 9. A function $f : (X, m_X) \to (Y, m_Y)$ is said to has a strongly M-closed graph [25] if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exists an m_X-open set U containing x and an m_Y-open set V containing y such that $(U \times m_Y\text{-}Cl(V)) \cap G(f) = \varnothing$.

Lemma 18. [25] *A function $f : (X, m_X) \to (Y, m_Y)$ has a strongly M-closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exists an m_X-open set U containing x and m_Y-open set V containing y such that $f(U) \cap m_Y\text{-}Cl(V) = \varnothing$.*

Definition 10. A graph $G(f)$ of a function $f : X \to Y$ is said to be *co*-βR-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exists an clopen set U in X containing x and $V \in \beta R(Y, y)$ such that $(U \times V) \cap G(f) = \varnothing$.

Remark 3. *If a function $f : (X, m_X) \to (Y, m_Y)$ has the strongly M-closed graph, then for the special case $m_X = \tau_q$ and $m_Y = \beta O(Y)$, $G(f)$ has co-βR-closed graph and we may state the following theorem.*

Theorem 19. *The following properties are equivalent for a graph $G(f)$ of a function:*

(a) *$G(f)$ is co-βR-closed.*

(b) *for each $(x, y) \in (X \times Y) - G(f)$, there exists a clopen set U containing x in X and $V \in \beta R(Y, y)$ such that $f(U) \cap V = \varnothing$.*

(c) *for each point $(x, y) \in (X \times Y) - G(f)$, there exists a clopen set U containing x in X and $V \in \beta O(Y, y)$ such that. $f(U) \cap \beta cl(V) = \varnothing$.*

(d) *for each point $(x, y) \in (X \times Y) - G(f)$, there exists a quasi-open set U containing x in X and $V \in \beta O(Y, y)$ such that. $f(U) \cap \beta cl(V) = \varnothing$.*

Theorem 20. *If $f : X \to Y$ is weakly (τ, β)-continuous function and Y is β-T_2 , then $G(f)$ is co-βR-closed. in $X \times Y$.*

Proof. First suppose Y is β-T_2. Let $(x, y) \in (X \times Y) - G(f)$. It follows that $f(x) \neq y$. Since Y is β-T_2, there exists $V \in \beta O(Y, f(x))$ and $W \in \beta O(Y, y)$ such that $\beta cl(V) \cap \beta cl(W) = \varnothing$. Since f is weakly (τ, β)-continuous, there exists a clopen set $U = f^{-1}(\beta cl(V))$ in X containing x such that $f(U) \subset \beta cl(V)$. Therefore $f(U) \cap \beta cl(W) = \varnothing$ and $G(f)$ is *co*-βR-closed with respect to $X \times Y$. □

Theorem 21. *Let $f : X \to Y$ have a co-βR-closed graph. Then the following properties hold:*

(a) *if f is injective then X is ultra Hausdorff;*

(b) *if f is surjective then X is β-T_2.*

Proof. (a) Suppose that x and y are any two distinct points of X by the injectivity of f, $(x, f(y)) \notin G(f)$. Since $G(f)$ is *co*-βR-closed, by Theorem 19, there exist a clopen set U containing x and $V \in \beta O(Y, f(y))$ such that $f(U) \cap \beta cl(V) = \varnothing$. We have $U \cap$

$f^{-1}(\beta cl(V)) = \varnothing$. Therefore $y \notin U$. Then U and $X - U$ are disjoint clopen sets containing x and y, respectively. Hence X is ultra Hausdorff.

(b) Let y_1 and y_2 be any two distinct points of Y. Since f is surjective there exists a point $x \in X$ such that $f(x) = y_2$. Since $G(f)$ is *co*-βR-closed and $(x, y_1) \notin G(f)$ there exists a clopen set U containing x and $V \in \beta R(Y, y_1)$ such that $f(U) \cap V = \varnothing$. Therefore we have $y_2 \in f(U) \subset Y - V \in \beta R(Y)$ and hence Y is β-T_2. □

Theorem 22. *If $f : X \to Y$ is weakly (τ, β)-continuous, closed injection and Y is β-normal, then X is ultranormal.*

Proof. Suppose that A and B be any two disjoint closed subset of X. Since f is closed and injective $f(A)$ and $f(B)$ are disjoint closed subset of Y. Since Y is β-normal $f(A)$ and $f(B)$ can be separated by disjoint β-open sets. Hence there is a β-regular set W containing $f(A)$ and disjoint from $f(B)$. Since f is weakly (τ, β)-continuous, the inverse image of W under f is a clopen subset of X containing A and disjoint from B. Thus X is ultranormal. □

Definition 11. A space X is said to be

(a) β-regular [3] if for each closed set F and each point $x \in X - F$, there exist disjoint β-open sets U and V such that $x \in U$ and $F \subset V$,

(b) β^*-regular if for each β-closed set A of X and $x \in X$ such that $x \notin A$, there exist disjoint β-open sets U and V such that $x \in U$ and $A \subseteq V$,

(c) almost β-regular if for any regular closed set $F \subset X$ and each point $x \in X - F$, there exist disjoint β-open sets U and V such that $x \in U$ and $F \subset V$.

Theorem 23. *The following properties hold for a function $f : X \to Y$:*

(a) *If X is a β^*-regular space and f is weakly (τ, β)-continuous, then f is θ-β-irresolute.*

(b) *If $f : X \to Y$ is weakly (τ, β)-continuous and Y is a β-regular space, then f is clopen continuous.*

(c) *If $f : X \to Y$ is weakly (τ, β)-continuous and X is a almost β-regular space, then f is weakly β-irresolute*

Proof. (a) Let $x \in X$ and $V \in \beta O(Y, f(x))$. Since f is weakly (τ, β)-continuous, there exists $U \in O(X, x)$ such that $f(U) \subseteq \beta cl(V)$. Since X is β^*-regular there exists $W \in \beta O(X, x)$ such that $\beta cl(W) \subseteq U$.Therefore we obtain $f(\beta cl(W)) \subseteq \beta cl(V)$. This shows that f is θ-β-irresolute.

(b) Let $x \in X$ and V be any open subset of Y containing $f(x)$. Then there exists $W \in \beta O(Y, f(x))$ such that $\beta cl(W) \subset V$. Since $\beta cl(W) \in \beta R(Y, f(x))$ by Theorem 4, there exists a clopen set U containing x such that $f(U) \subset \beta cl(W) \subset V$. This shows that f is clopen continuous.

(c) Let $x \in X$ and $V \in \beta O(Y, f(x))$. Since f is weakly (τ, β)-continuous, there exists regular open set U such that $f(U) \subseteq \beta cl(V)$. Then there exists $W \in \beta O(X, x)$ such that $x \in W \subset \beta cl(W) \subset U$. Then $f(W) \subset \beta cl(V) \subset V$. This shows that f is weakly β-irresolute. □

Comparisons

Definition 12. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(a) almost s-continuous [22] (resp. weakly (τ, β)-continuous [24], $p(\theta)$-continuous [6]) if for each $x \in X$ and each semiopen (β-open, preopen) set V containing $f(x)$, there exists an open set U containing $f(x)$ such that $f(U) \subset scl(V)$ (resp. $f(U) \subset \beta cl(V)$, $f(U) \subset pcl(V)$),

(b) strongly β-irresolute function [21] if for each $x \in X$ and each β-open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset V$,

(c) completely β-irresolute [37] (resp. perfectly β-irresolute [37]) if $f^{-1}(V)$ is regular open (clopen) in X for every β-open set V of Y,

(d) almost clopen [10] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists a clopen set U containing x such that $f(U) \subset int(cl(V))$,

(e) regular set-connected [9] if the preimage of every regular open subset of Y is clopen in X.

Remark 4. *We have the following implications for a function $f : X \to Y$:*

Perfectly β-irresolute⇒completely β-irresolute⇒strongly β-irresolute function ⇒weakly (τ, β)-continuous⇒almost s-continuous⇒regular set connected⇒almost clopen⇒almost continuous.

Remark 5. *None of these implications is not reversible in general as related articles* [10,24,37] *shows.*

Covering Properties

Definition 13. A subset K of a nonempty set X with a minimal structure m_X is said to be m-compact [24] (m-closed [24]) relative to (X, m_X) if any cover $\{U_i : i \in I\}$ of K by m_X-open sets, there exists a finite subset I_0 of I such that $K \subseteq \cup\{U_i : i \in I_0\}$ ($K \subseteq \cup\{m_X\text{-}Cl(U_i) : i \in I_0\}$). (X, m_X) is m-closed if X is m-closed relative to (X, m_X).

Definition 14. *A subset K of a space X is said to be β-closed [2] (resp. mildly compact [33], quasi H-closed [31]) relative to X if for every cover $\{V_\alpha : \alpha \in I\}$ of K by β-open (resp. clopen, open) subsets of X, there exists a finite subset I_0 of I such that $K \subset \cup\{\beta cl(V_\alpha) : \alpha \in I_0\}$ (resp. $K \subset \cup\{V_\alpha : \alpha \in I_0\}$, $K \subset \cup\{cl(V_\alpha) : \alpha \in I_0\}$).*

Definition 15. A topological space (X, τ) is said to be countably β-closed if every countable cover of X by β-open sets has a finite subcover whose β-closures cover X.

Remark 6. *For a subset A of a space X, A is β-closed relative to X if and only if every cover of A by β-regular subsets of X has a finite subcover.*

Theorem 24. *If a function $f : X \to Y$ is weakly (τ, β)-continuous and K is mildly compact relative to X, then $f(K)$ is β-closed relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be any cover of $f(K)$ by β-regular sets of Y. By Theorem 2, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of K by clopen subsets of X. Therefore there exists a finite subset I_0 of I such that $K \subset \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. It follows from Remark 6 that $f(K)$ is β-closed. □

Corollary 25. *If a function $f : X \to Y$ is weakly (τ, β)-continuous and K is quasi H-closed relative to X, then $f(K)$ is β-closed relative to Y.*

Remark 7. *A topological space (X, τ) is countably β-closed relative to X if and only if every countable cover of X by β-regular sets has a finite subcover.*

Definition 16. A topological space (X, τ) is called mildly countably compact [33] if every countable clopen cover of X admits a finite subcover.

Theorem 26. *Let $f : (X, \tau) \to (Y, \sigma)$ be a weakly (τ, β)-continuous surjection.*

(a) *If X is mildly compact, then Y is β-closed.*

(b) *If X is mildly countably compact, then Y is countably β-closed.*

Proof. (a) Let $\{V_i : i \in I\}$ be a cover of Y consisting of β-regular sets. Since f is weakly (τ, β)-continuous and onto, then by Theorem 2 each one of the sets $U_i = f^{-1}(V_i)$ is clopen in X. Since $\{U_i : i \in I\}$ is a clopen cover of X and since X is mildly compact, then for some finite $I_0 \subseteq I$, we have $X = \cup_{i \in I_0} U_i$. Thus $Y = \cup_{i \in I_0} V_i$, which shows that Y is β-closed.

(b) Similar to (a). □

Theorem 27. [20] *Let $f : (X, m_X) \to (Y, m_Y)$ be a function. Assume that m_X is a base for a topology. If the graph $G(f)$ is strongly M-closed, then m_X-$Cl(f^{-1}(K)) = f^{-1}(K)$ whenever the set $K \subseteq Y$ is m_Y-closed relative to (Y, m_Y).*

Corollary 28. *If a function $f : (X, \tau_q) \to (Y, m_Y)$ has a strongly M-closed graph, then $f^{-1}(K)$ is quasi-closed in (X, τ_q) for each set K which is m_Y-closed relative to (Y, m_Y).*

Corollary 29. *If a function $f : X \to Y$ has co-βR-closed graph, then $f^{-1}(K)$ is quasi-closed in X for every subset K which is β-closed relative to Y.*

Theorem 30. *If a function $f : X \to Y$ has a co-βR-closed graph and Y is β-closed then f is weakly (τ, β)-continuous.*

Proof. Let $V \in \beta R(Y)$, then by Lemma 1, $Y - V \in \beta R(Y)$. By the β-closedness of Y, $Y - V$ is β-closed. By Corollary 29, $f^{-1}(Y - V) = X - f^{-1}(V)$ is quasi-closed, hence $f^{-1}(V)$ is quasi open. Set $U = f^{-1}(V)$, then $f(U) \subset V$, and by Theorem 4, f is weakly (τ, β)-continuous. □

Corollary 31. *Let Y be a β-closed β-T_2 space. The following are equivalent for a function $f : X \to Y$:*

(a) f *is weakly* (τ,β)*-continuous.;*

(b) $G(f)$ *is co-*βR*-closed;*

(c) *for each* K*,* β*-closed relative to* Y*,* $f^{-1}(K)$ *is quasi-closed in* X*.*

Proof. This is a direct consequence of Corollary 29 and Theorems 20 and 30. □

Conclusion. *Weak* (τ,β)*-continuity is closely related with clopen sets and has similar properties with weakly clopen functions* [35]. *Then it is plausible to study a new function type as a new form of weak* M*-continuity from a space with quasi-topology* τ_q*, to a space with an* m*-structure, that is a function* $f : (X,\tau_q) \to (Y, m_Y)$ *which can be named as weakly* (τ_q, m)*-continuous functions.*

Acknowledgement. The author wishes to thank Professor Valeriu Popa from the Department of Mathematics and Informatics, Faculty of Sciences, "Vasile Alecsandri" University of Bacau, Romania for valuable comments on an earlier version of this paper.

References

1. Abd El-Monsef M.E., El-Deeb S.N., Mahmoud R.A., β-open sets and β-continuous mappings. *Bulletin of the Faculty of Science. A. Physics and Mathematics, Assiut University,* 12, pp. 77–90, 1983.

2. Abd El-Monsef M.E., Kozae A.M., Some generalized forms of compactness and closedness, *Delta Journal of Science*, 9 (2), pp. 257–269, 1985.

3. Abd El-Monsef M.E., Geaisa A.N., Mahmoud R.A., β-regular spaces, *Proceedings of the Mathematical and Physical Society of Egypt*, 60, pp. 47-52, 1985.

4. Andrijevic D., Semi-preopen sets, *Matematichki Vesnik*, 38, pp. 24–32, 1986.

5. Basu C. K., Ghosh M.K., β-closed spaces and β-θ subclosed graphs, *European Journal of Pure and Applied Mathematics*, 1, No. 3, pp. 40-50, 2008.

6. Debray A., Investigations of some properties of topology and certain allied structure, *Ph.D.Thesis*, *University of Calcutta*, 1999.

7. Cho S.H., A note on almost s-continuous functions, *Kyungpook Mathematical Journal,* 42, No.1, pp. 171-175, 2002.

8. Di Concilio A., On θ-continuous convergence in function spaces, *Rendiconti di Matematica e delle Sue Applicazioni. Serie VII, 4*, pp. 85-94, 1984.

9. Dontchev J., Ganster M., Reilly I.L., More on almost s-continuity, *Indian Journal of Mathematics,* 41 , pp. 139–146, 1999.

10. Ekici E., Generalization of perfectly continuous, regular set-connected and clopen functions, *Acta Mathematica Hungarica,* 107, No. 3, pp. 193-206, 2005.

11. Georgiou D.N., Topologies on function spaces, *Rendiconti Del Circolo Matematico Di Palermo Serie II,* 52, No. 1, pp. 145-157, 2003.

12. Jafari S., Noiri T., Some properties of almost s-continuous functions, *Rendiconti Del Circolo Matematico Di Palermo Serie II*, Tomo XLVIII, pp. 571-582, 1999.

13. Jafari S., Noiri T., Properties of β-connected spaces, *Acta Mathematica Hungarica,* 101, No. 3, 227-236, 2003.

14. Jain R.C., The Role of regularly Open Sets in General Topology, Ph.D. Thesis, Meerut University, Institute of Advanced Studies, 1980.

15. Kohli J. K., Singh D., Almost cl-supercontinuous functions, *Applied General Topology*, 10 No 1 , pp. 1-12, 2009.

16. Levine N., Semi-open sets and semi-continuity in topological spaces, *American Mathematical Monthly*, 70 , 36-41, 1963.

17. Long P., Herrington L., The T_θ-topology and faintly continuous functions, *Kyungpook Mathematical Journal,* 22, pp. 7-14, 1982.

18. Maki H., Rao K.C., Gani N., On generalizing semi-open and preopen sets, *Pure & Applied Mathematika Sciences*, 49,17-29, l999.

19. Mashhour A.S., Abd El-Monsef M.E., El-Deeb S. N., On precontinuous and weak precontinuous functions,*Proceedings of the Mathematical and Physical Society of Egypt,* 51, pp. 47–53, 1982.

20. Mocanu M., On m-compact spaces, *Rendiconti Del Circolo Matematico Di Palermo Serie II,* Tomo LIV , pp. 119-144, 2005.

21. Nasef A.A., Noiri T., Strongly β-irresolute functions, *Journal of Natural Sciences and Mathematics,* 36, No.2, pp. 199-206, 1996.

22. Noiri T., Ahmad M. B., Khan M., Almost s-continuous functions, *Kyungpook Mathematical Journal,* Vol. 35 , pp. 311-322, 1995.

23. Noiri T., Weakly α-continuous functions., *International Journal of Mathematics and Mathematical Sciences,* 10, pp. 483-490, 1987.

24. Noiri T., Popa V., On weakly (τ, m)-continuous functions, *Rendiconti Del Circolo Matematico Di Palermo Serie II*, 51, No. 2, pp. 295-316, 2002.

25. Noiri T., Popa V., A unified theory of weak continuity for functions, *Rendiconti Del Circolo Matematico Di Palermo Serie II*, 51, No. 3, pp. 439-464, 2002 .

26. Noiri T., Popa V., Minimal structure,weakly irresolute functions and bitopological spaces, *Studii şi Cercetări Ştiinţifice, Seria Matematică, University of Bacău*, 18 (2008), 181-192.

27. Noiri T., Weak and strong forms of β-irresolute functions, *Acta Mathematica Hungarica,* 99, No. 4, 315-328, 2003.

28. Njastad O., On some classes of nearly open sets, *Pacific Journal of Mathematics,* 15, 961–970, 1965.

29. Son M.J., Park J.H., Lim K.M., Weakly clopen functions, *Chaos, Solitons & Fractals*, 33, 1746-1755, 2007.

30. Popa V., Noiri T., On M-continuous functions, *Annals of the. "Dunarea de Jos" University of Galaţi, Mathematics Physics, Theoretical Mechanics, Fascicle II,* 18 (23), pp. 31–41, 2000.

31. Porter J., Thomas J., On H-closed and minimal Hausdorff spaces. *Transactions of the American Mathematical Society,* 138, 159-170, 1969.

32. Singal M.K., Singal A.R., Almost-continuous mappings, *Yokohama Mathematical Journal*, 16, 63–73, 1968.

33. Staum R., The algebra of bounded continuous functions into a nonarchimedean field, *Pacific Journal of Mathematics,* 50, 169-185, 1974.

34. Stone M.H., Applications of the theory of Boolean rings to general topology, *Transactions of the American Mathematical Society,* 41, 375–381, 1937.

35. Tahiliani S., Generalized β-closed functions, *Bulletin of the Calcutta Mathematical Society,* 98, No. 4, pp. 367-376, 2006.

36. Veličko N.V., H-closed topological spaces, *American Mathematical Society Translations*, 78, pp. 103-118, 1968.

37. Zorlutuna İ., On strong forms of completely irresolute functions, *Chaos, Solitons & Fractals* 38, pp. 970-979, 2008.

Proprietati ale functiilor slab (τ, β)-continue

Rezumat

Notiunea de functii slab (τ, β)-continue a fost introdusa de Popa si Noiri. In aceasta lucrare obtinem cateva proprietati si o caracterizare noua a functiilor slab (τ, β)-continue si aratam ca multe dintre rezultatele cunoscute pot fi intarite. Deci imbunatatim si intarim cateva dintre rezultatele referitoare la functii slab (τ, β)-continue.