









By (4) and (6), we get

$$z = \lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} By_n. \quad (7)$$

(1) Suppose that  $A(X)$  is a closed subspace of  $(X, d)$ . Then  $z \in A(X)$ . Since  $AX \subseteq TX$ , then there exists  $u \in X$  such that  $z = Tu$ . By (H 3), we get

$$d(Ax_{2n}, Bu) \leq k[d(Sx_{2n}, Tu) + d(Ax_{2n}, Sx_{2n}) + d(Bu, Tu) + d(Sx_{2n}, Bu) + d(Ax_{2n}, Tu)],$$

which, by letting  $n \rightarrow \infty$ , implies that

$$d(z, Bu) \leq 2kd(z, Bu). \quad (8)$$

Since  $k < \frac{1}{3}$ , then it follows from (8) that  $z = Bu$ . Thus, we have  $z = Tu = Bu$ .

Since  $B(X) \subset S(X)$ , then there exists  $v \in X$  such that  $Bu = Sv$ . Then  $z = Tu = Bu = Sv$ . By applying the inequality (H 3), we get

$$\begin{aligned} d(Av, Sv) &= d(Av, Bu) \\ &\leq k[d(Sv, Tu) + d(Av, Sv) + d(Bu, Tu) + d(Sv, Bu) + d(Av, Tu)] \\ &= 2kd(Av, Sv), \end{aligned}$$

which implies that  $Av = Sv$ . Hence, we obtain

$$z = Tu = Bu = Sv = Av. \quad (9)$$

The conclusions in (9) will be obtained by similar arguments, if we suppose that  $T(X)$ ,  $B(X)$  or  $S(X)$  is a closed subspace of  $X$ .

(2) By (H 3) it follows that  $z$  (given in (9)) is the unique point of coincidence for  $(A, S)$  and for  $(B, T)$ . By Lemma 1. of G. Jungck and B.E. Rhoades, we conclude that  $z$  is the unique common fixed point of  $A, B, S$  and  $T$ .

(II) If we suppose that the pair  $\{B, T\}$  satisfies the property (E.A), then by similar arguments we obtain the same conclusions as in the part (I).

(III) It remains to show the uniqueness of the fixed common fixed point  $z$ . Suppose that  $w$  is another common fixed point for the mappings  $A, B, S$  and  $T$ , such that  $w \neq z$ . Obviously we have  $\sigma(w, z) = 3d(w, z) > 0$ . Then, by applying the condition (H 3), we obtain

$$d(w, z) = d(Aw, Bz) \leq k\sigma(w, z) = 3kd(w, z),$$

which is a contradiction. So the mappings  $A, B, S$  and  $T$  have a unique common fixed point. This completes the proof.

As a consequence, we have the following.

**Corollary 1.** Let  $(A, S)$  and  $(B, T)$  be two occasionally weakly compatible pairs of self-mappings of a metric space  $(X, d)$  such that

$$(H1) : AX \subseteq TX \text{ and } BX \subseteq SX,$$

$$(H2) : \text{one of } AX, BX, SX \text{ or } TX \text{ is a closed subspace of } (X, d),$$

$$(H3) : d(Ax, By) \leq k \sigma(x, y), \text{ for all } x, y \in X, \text{ where } k \text{ is such that } 0 \leq k < \frac{1}{3}.$$

If one of the following two conditions is satisfied.

(i)  $A$  and  $S$  are noncompatible, or

(ii)  $B$  and  $T$  are noncompatible.

Then the mappings  $A, B, S$  and  $T$  have a unique common fixed point.

## Remarks

**Remarks.** We observe that in both Theorem 1 and Theorem 2 the condition (iii) seems to be incorrect. The symbol " $<$ " used in this condition leads to a contradiction. Indeed, the existence of a common fixed point  $z$  in  $X$  (as asserted in both these theorems) would imply that  $0 < 0$ , a contradiction.

The author thinks that, it would be more convenient to replace the condition (iii) by the condition (H 3) as given in the main result of this paper to avoid contradiction. So our Theorem 4 provides a correction and some improvements to Theorem 1 and to Theorem 2.

By using a result of J. Jachymski [4], it is easy to see that the conditions (a), (c), (c') of Theorem 3 imply that the pairs  $(A, S)$  and  $(B, T)$  satisfy the property (E.A). Thus we can obtain Theorem 3. as a consequence of our Theorem 4.

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## **O teoremă de punct fix comun pentru două perechi de multifuncții care satisfac proprietatea (E.A)**

### **Rezumat**

*În 2003, K. Jha, R. P. Pant și S. L. Singh au demonstrat în [5] o teoremă de punct fix comun pentru două perechi de aplicații compatibile care satisfac o condiție contractivă de tip Meir-Keeler și o condiție de tip Lipschitz. În 2008, această teoremă a fost extinsă de H. Bouhadjera și A. Djoudi (vezi [3]) la două perechi de aplicații slab compatibile fără a folosi continuitatea. Scopul acestei lucrări este extinderea rezultatelor din [5], [3] și alte lucrări la cazul a două perechi de aplicații ocazional slab compatibile, dintre care una satisface condiția (E.A). Eliminăm condiția de tip Meir-Keeler și păstrăm numai condiția de tip Lipschitz, care pentru constante Lipschitz  $k \geq 1/5$  nu mai este o condiție contractivă de tip clasic. Abordarea noastră permite obținerea unor rezultate noi în teoria punctelor fixe în spații metrice.*